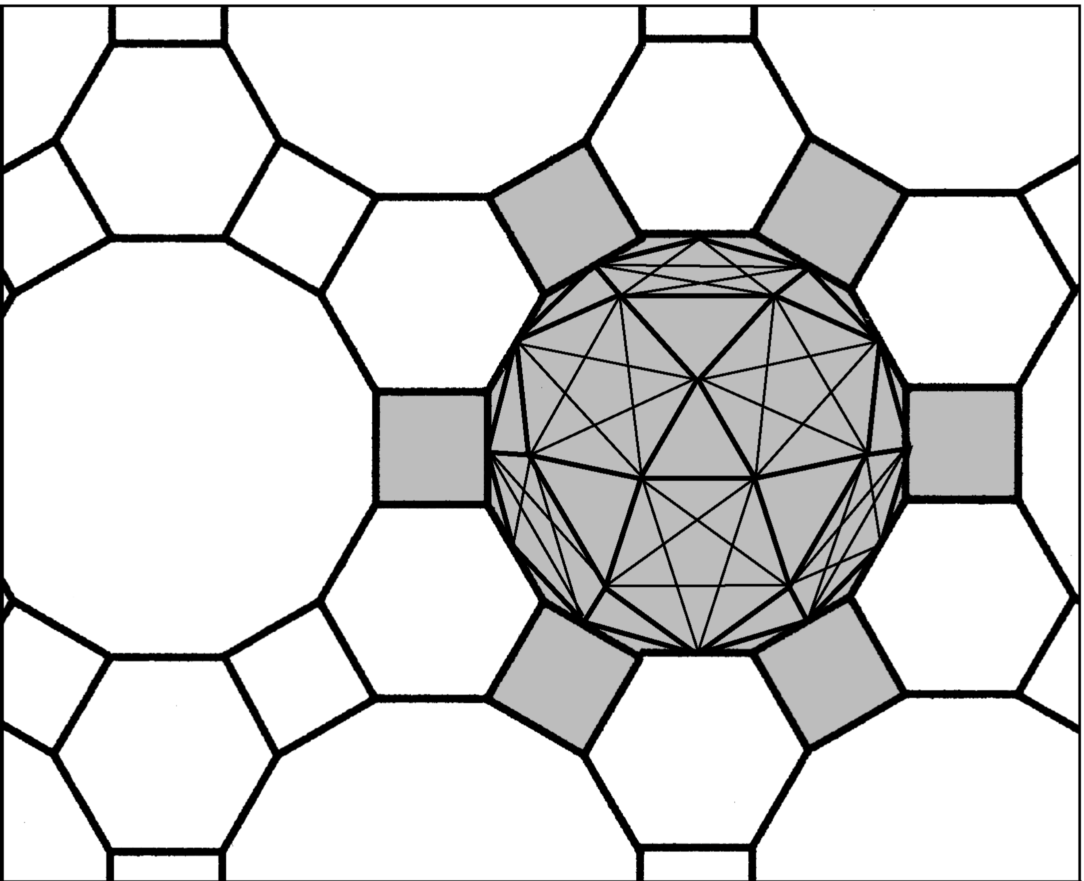
Workbook 3



Equations, Inequalities & Partial Fractions

HELM

HELM: Helping Engineers Learn Mathematics

http://helm.lboro.ac.uk

About the HELM Project

HELM (Helping Engineers Learn Mathematics) materials were the outcome of a three-year curriculum development project undertaken by a consortium of five English universities led by Loughborough University, funded by the Higher Education Funding Council for England under the Fund for the Development of Teaching and Learning for the period October 2002 – September 2005, with additional transferability funding October 2005 – September 2006.

HELM aims to enhance the mathematical education of engineering undergraduates through flexible learning resources, mainly these Workbooks.

HELM learning resources were produced primarily by teams of writers at six universities: Hull, Loughborough, Manchester, Newcastle, Reading, Sunderland.

HELM gratefully acknowledges the valuable support of colleagues at the following universities and colleges involved in the critical reading, trialling, enhancement and revision of the learning materials:
Aston, Bournemouth & Poole College, Cambridge, City, Glamorgan, Glasgow, Glasgow Caledonian, Glenrothes Institute of Applied Technology, Harper Adams, Hertfordshire, Leicester, Liverpool, London Metropolitan, Moray College, Northumbria, Nottingham, Nottingham Trent, Oxford Brookes, Plymouth, Portsmouth, Queens Belfast, Robert Gordon, Royal Forest of Dean College, Salford, Sligo Institute of Technology, Southampton, Southampton Institute, Surrey, Teesside, Ulster, University of Wales Institute Cardiff, West Kingsway College (London), West Notts College.

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Production of this 2015 edition, containing corrections and minor revisions of the 2008 edition, was funded by the **sigma** Network.

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Contents 3

Equations, Inequalities & Partial Fractions

Learning outcomes

In this Workbook you will learn about solving single equations, mainly linear and quadratic, but also cubic and higher degree, and also simultaneous linear equations. Such equations often arise as part of a more complicated problem. In order to gain confidence in mathematics you will need to be thoroughly familiar with these basis topics.
You will also study how to manipulate inequalities. You will also be introduced to partial fractions which will enable you to re-express an algebraic fraction in terms of simpler fractions. This will prove to be extremely useful in later studies on integration.

Solving Linear Equations

3.1

Introduction

Many problems in engineering reduce to the solution of an equation or a set of equations. An equation is a type of mathematical expression which contains one or more unknown quantities which you will be required to find. In this Section we consider a particular type of equation which contains a single unknown quantity, and is known as a linear equation. Later Sections will describe techniques for solving other types of equations.

Prerequisites

Before starting this Section you should . . .

- be able to add, subtract, multiply and divide fractions
- be able to transpose formulae

Learning Outcomes

On completion you should be able to . . .

- recognise and solve a linear equation

1. Linear equations

Key Point 1

A **linear equation** is an equation of the form

$$ax + b = 0 \qquad a \neq 0$$

where a and b are known numbers and x represents an unknown quantity to be found.

In the equation $ax + b = 0$, the number a is called the **coefficient of** x, and the number b is called the **constant term**.

The following are examples of linear equations

$$3x + 4 = 0, \quad -2x + 3 = 0, \quad -\frac{1}{2}x - 3 = 0$$

Note that the unknown, x, appears only to the first power, that is as x, and not as x^2, $\sqrt{x}$, $x^{1/2}$ etc. Linear equations often appear in a non-standard form, and also different letters are sometimes used for the unknown quantity. For example

$$2x = x + 1 \qquad 3t - 7 = 17, \qquad 13 = 3z + 1, \qquad 1 - \frac{1}{2}y = 3 \qquad 2\alpha - 1.5 = 0$$

are all examples of linear equations. Where necessary the equations can be rearranged and written in the form $ax + b = 0$. We will explain how to do this later in this Section.

Task

Which of the following are linear equations and which are not linear?

(a) $3x + 7 = 0$, (b) $-3t + 17 = 0$, (c) $3x^2 + 7 = 0$, (d) $5p = 0$

The equations which can be written in the form $ax + b = 0$ are linear.

Your solution

(a) (b) (c) (d)

Answer

(a) linear in x (b) linear in t (c) non-linear - quadratic in x (d) linear in p, constant is zero

To solve a linear equation means to find the value of x that can be substituted into the equation so that the left-hand side equals the right-hand side. Any such value obtained is known as a **solution** or **root** of the equation and the value of x is said to **satisfy** the equation.

Example 1

Consider the linear equation $3x - 2 = 10$.

(a) Check that $x = 4$ is a solution.

(b) Check that $x = 2$ is *not* a solution.

Solution

(a) To check that $x = 4$ is a solution we substitute the value for x and see if both sides of the equation are equal. Evaluating the left-hand side we find $3(4) - 2$ which equals 10, the same as the right-hand side. So, $x = 4$ is a solution. We say that $x = 4$ satisfies the equation.

(b) Substituting $x = 2$ into the left-hand side we find $3(2) - 2$ which equals 4. Clearly the left-hand side is not equal to 10 and so $x = 2$ is not a solution. The number $x = 2$ does not satisfy the equation.

Task

Test which of the given values are solutions of the equation

$$18 - 4x = 26$$

(a) $x = 2$, (b) $x = -2$, (c) $x = 8$

(a) Substituting $x = 2$, the left-hand side equals

Your solution

Answer

$18 - 4 \times 2 = 10$. But $10 \neq 26$ so $x = 2$ is not a solution.

(b) Substituting $x = -2$, the left-hand side equals:

Your solution

Answer

$18 - 4(-2) = 26$. This is the same as the right-hand side, so $x = -2$ is a solution.

(c) Substituting $x = 8$, the left-hand side equals:

Your solution

Answer

$18 - 4(8) = -14$. But $-14 \neq 26$ and so $x = 8$ is not a solution.

Exercises

1. (a) Write down the general form of a linear equation.
   (b) Explain what is meant by the root or solution of a linear equation.

In questions 2-8 verify that the given value is a solution of the given equation.

2. $3z - 7 = -28, \quad z = -7$
3. $8x - 3 = -11, \quad x = -1$
4. $2s + 3 = 4, \quad s = \frac{1}{2}$
5. $\frac{1}{3}x + \frac{4}{3} = 2, \quad x = 2$
6. $7t + 7 = 7, \quad t = 0$
7. $11x - 1 = 10, \quad x = 1$
8. $0.01t - 1 = 0, \quad t = 100.$

Answers

1. (a) The general form is $ax + b = 0$ where a and b are known numbers and x represents the unknown quantity.
   (b) A root is a value for the unknown which satisfies the equation.

2. Solving a linear equation

To solve a linear equation we make the unknown quantity the **subject** of the equation. We obtain the unknown quantity on its own on the left-hand side. To do this we may apply the same rules used for transposing formulae given in Workbook 1 Section 1.7. These are given again here.

Key Point 2

Operations which can be used in the process of solving a linear equation

- add the same quantity to both sides
- subtract the same quantity from both sides
- multiply both sides by the same quantity
- divide both sides by the same quantity
- take the reciprocal of both sides (invert)
- take functions of both sides; for example cube both sides.

A useful summary of the rules in Key Point 2 is 'whatever we do to one side of an equation we must also do to the other'.

Example 2

Solve the equation $x + 14 = 5$.

Solution

Note that by subtracting 14 from both sides, we leave x on its own on the left. Thus

$$\begin{aligned} x + 14 - 14 &= 5 - 14 \\ x &= -9 \end{aligned}$$

Hence the solution of the equation is $x = -9$. It is easy to check that this solution is correct by substituting $x = -9$ into the original equation and checking that both sides are indeed the same. You should get into the habit of doing this.

Example 3

Solve the equation $19y = 38$.

Solution

In order to make y the subject of the equation we can divide both sides by 19:

$$\begin{aligned} 19y &= 38 \\ \frac{19y}{19} &= \frac{38}{19} \\ \text{cancelling 19's gives} \quad y &= \frac{38}{19} \\ \text{so} \quad y &= 2 \end{aligned}$$

Hence the solution of the equation is $y = 2$.

Example 4

Solve the equation $4x + 12 = 0$.

Solution

Starting from $4x + 12 = 0$ we can subtract 12 from both sides to obtain

$$\begin{aligned} 4x + 12 - 12 &= 0 - 12 \\ \text{so that} \qquad 4x &= -12 \end{aligned}$$

If we now divide both sides by 4 we find

$$\begin{aligned} \frac{4x}{4} &= \frac{-12}{4} \\ \text{cancelling 4's gives} \qquad x &= -3 \end{aligned}$$

So the solution is $x = -3$.

Task

Solve the linear equation $14t - 56 = 0$.

Your solution

Answer

$t = 4$

Example 5

Solve the following equations: (a) $x + 3 = \sqrt{7}$, (b) $x + 3 = -\sqrt{7}$.

Solution

(a) Subtracting 3 from both sides gives $x = \sqrt{7} - 3$.

(b) Subtracting 3 from both sides gives $x = -\sqrt{7} - 3$.

Note that when asked to solve $x + 3 = \pm\sqrt{7}$ we can write the two solutions as $x = -3 \pm \sqrt{7}$. It is usually acceptable to leave the solutions in this form (i.e. with the $\sqrt{7}$ term) rather than calculate decimal approximations. This form is known as the **surd form**.

Example 6

Solve the equation $\frac{2}{3}(t+7) = 5$.

Solution

There are a number of ways in which the solution can be obtained. The idea is to gradually remove unwanted terms on the left-hand side to leave t on its own. By multiplying both sides by $\frac{3}{2}$ we find

$\frac{3}{2} \times \frac{2}{3}(t+7) = \frac{3}{2} \times 5 = \frac{3}{2} \times \frac{5}{1}$ and after simplifying and cancelling, $\quad t+7 = \frac{15}{2}$

Finally, subtracting 7 from both sides gives

$$t = \frac{15}{2} - 7 = \frac{15}{2} - \frac{14}{2} = \frac{1}{2}$$

So the solution is $t = \frac{1}{2}$.

Example 7

Solve the equation $3(p-2) + 2(p+4) = 5$.

Solution

At first sight this may not appear to be in the form of a linear equation. Some preliminary work is necessary. Removing the brackets and collecting like terms we find the left-hand side yields $5p+2$ so the equation is $5p+2 = 5$ so that $p = \frac{3}{5}$.

Task

Solve the equation $2(x-5) = 3 - (x+6)$.

(a) First remove the brackets on both sides:

Your solution

Answer

$2x - 10 = 3 - x - 6$. We may write this as $2x - 10 = -x - 3$.

(b) Rearrange the equation found in (a) so that terms involving x appear only on the left-hand side, and constants on the right. Start by adding 10 to both sides:

Your solution

Answer

$2x = -x + 7$

(c) Now add x to both sides:

Your solution

Answer

$3x = 7$

(d) Finally solve this to find x:

Your solution

$x =$

Answer

$\frac{7}{3}$

Example 8

Solve the equation

$$\frac{6}{1-2x} = \frac{7}{x-2}$$

Solution

This equation appears in an unfamiliar form but it can be rearranged into the standard form of a linear equation. By multiplying both sides by $(1-2x)$ and $(x-2)$ we find

$$(1-2x)(x-2) \times \frac{6}{1-2x} = (1-2x)(x-2) \times \frac{7}{x-2}$$

Considering each side in turn and cancelling common factors:

$$6(x-2) = 7(1-2x)$$

Removing the brackets and rearranging to find x we have

$$6x - 12 = 7 - 14x$$

Further rearrangement gives: $20x = 19$

The solution is therefore $x = \dfrac{19}{20}$.

Example 9

Figure 1 shows three branches of an electrical circuit which meet together at x. Point x is known as a **node**. As shown in Figure 1 the current in each of the branches is denoted by I, I_1 and I_2. Kirchhoff's current law states that the current entering any node must equal the current leaving that node. Thus we have the equation $I = I_1 + I_2$

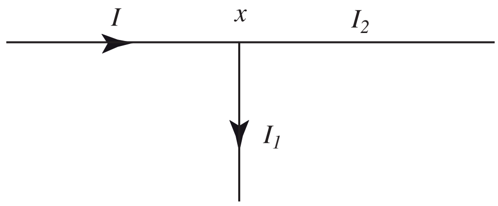


Figure 1

(a) Given $I_2 = 10$ A and $I = 18$ A calculate I_1.

(b) Suppose $I = 36$ A and it is known that current I_2 is five times as great as I_1. Find the branch currents.

Solution

(a) Substituting the given values into the equation we find $18 = I_1 + 10$.

Solving for I_1 we find

$$I_1 = 18 - 10 = 8$$

Thus I_1 equals 8 A.

(b) From Kirchhoff's law, $I = I_1 + I_2$.

We are told that I_2 is five times as great as I_1, and so we can write $I_2 = 5I_1$.

Since $I = 36$ we have

$$36 = I_1 + 5I_1$$

Solving this linear equation $36 = 6I_1$ gives $I_1 = 6$ A.

Finally, since I_2 is five times as great as I_1, we have $I_2 = 5I_1 = 30$ A.

Exercises

In questions 1-24 solve each equation:

1. $7x = 14$
2. $-3x = 6$
3. $\frac{1}{2}x = 7$
4. $3x = \frac{1}{2}$
5. $4t = -2$
6. $2t = 4$
7. $4t = 2$
8. $2t = -4$
9. $\dfrac{x}{6} = 3$
10. $\dfrac{x}{6} = -3$
11. $7x + 2 = 9$
12. $7x + 2 = 23$
13. $-7x + 1 = -6$
14. $-7x + 1 = -13$
15. $\dfrac{17}{3}t = -2$
16. $3 - x = 2x + 8$
17. $x - 3 = 8 + 3x$
18. $\dfrac{x}{4} = 16$
19. $\dfrac{x}{9} = -2$
20. $-\dfrac{13}{2}x = 14$
21. $-2y = -6$
22. $-7y = 11$
23. $-69y = -690$
24. $-8 = -4\gamma.$

In questions 25-47 solve each equation:

25. $3y - 8 = \dfrac{1}{2}y$
26. $7t - 5 = 4t + 7$
27. $3x + 4 = 4x + 3$
28. $4 - 3x = 4x + 3$
29. $3x + 7 = 7x + 2$
30. $3(x + 7) = 7(x + 2)$
31. $2x - 1 = x - 3$
32. $2(x + 4) = 8$
33. $-2(x - 3) = 6$
34. $-2(x - 3) = -6$
35. $-3(3x - 1) = 2$
36. $2 - (2t + 1) = 4(t + 2)$
37. $5(m - 3) = 8$
38. $5m - 3 = 5(m - 3) + 2m$
39. $2(y + 1) = -8$
40. $17(x - 2) + 3(x - 1) = x$
41. $\dfrac{1}{3}(x + 3) = -9$
42. $\dfrac{3}{m} = 4$
43. $\dfrac{5}{m} = \dfrac{2}{m + 1}$
44. $-3x + 3 = 18$
45. $3x + 10 = 31$
46. $x + 4 = \sqrt{8}$
47. $x - 4 = \sqrt{23}$
48. If $y = 2$ find x if $4x + 3y = 9$
49. If $y = -2$ find x if $4x + 5y = 3$
50. If $y = 0$ find x if $-4x + 10y = -8$
51. If $x = -3$ find y if $2x + y = 8$
52. If $y = 10$ find x when $10x + 55y = 530$
53. If $\gamma = 2$ find β if $54 = \gamma - 4\beta$

In questions 54-63 solve each equation:

54. $\dfrac{x - 5}{2} - \dfrac{2x - 1}{3} = 6$
55. $\dfrac{x}{4} + \dfrac{3x}{2} - \dfrac{x}{6} = 1$
56. $\dfrac{x}{2} + \dfrac{4x}{3} = 2x - 7$
57. $\dfrac{5}{3m + 2} = \dfrac{2}{m + 1}$
58. $\dfrac{2}{3x - 2} = \dfrac{5}{x - 1}$
59. $\dfrac{x - 3}{x + 1} = 4$
60. $\dfrac{x + 1}{x - 3} = 4$
61. $\dfrac{y - 3}{y + 3} = \dfrac{2}{3}$
62. $\dfrac{4x + 5}{6} - \dfrac{2x - 1}{3} = x$
63. $\dfrac{3}{2s - 1} + \dfrac{1}{s + 1} = 0$
64. Solve the linear equation $ax + b = 0$ to find x
65. Solve the linear equation $\dfrac{1}{ax + b} = \dfrac{1}{cx + d}$ $(a \neq c)$ to find x

Answers

1. 2	2. -2	3. 14	4. $1/6$	5. $-1/2$	6. 2
7. $1/2$	8. -2	9. 18	10. -18	11. 1	12. 3
13. 1	14. 2	15. $-6/17$	16. $-5/3$	17. $-11/2$	18. 64
19. -18	20. $-28/13$	21. $y = 3$	22. $-11/7$	23. $y = 10$	24. 2
25. $16/5$	26. 4	27. 1	28. 1/7	29. $5/4$	30. 7/4
31. -2	32. 0	33. 0	34. 6	35. $1/9$	36. $-7/6$
37. $23/5$	38. 6	39. -5	40. 37/19	41. -30	42. 3/4
43. $-5/3$	44. -5	45. 7	46. $\sqrt{8} - 4$	47. $\sqrt{23} + 4$	48. 3/4
49. 13/4	50. 2	51. 14	52. -2	53. -13	54. -49
55. 12/19	56. 42	57. 1	58. 8/13	59. $-7/3$	60. 13/3
61. 15	62. 7/6	63. $-2/5$	64. $-b/a$	65. $\dfrac{(d-b)}{(a-c)}$	

Solving Quadratic Equations

3.2

Introduction

A **quadratic equation** is one which can be written in the form $ax^2 + bx + c = 0$ where a, b and c are numbers, $a \neq 0$, and x is the unknown whose value(s) we wish to find. In this Section we describe several ways in which quadratic equations can be solved.

Prerequisites

Before starting this Section you should ...

- be able to solve linear equations

Learning Outcomes

On completion you should be able to ...

- recognise a quadratic equation
- solve a quadratic equation by factorisation
- solve a quadratic equation using the standard formula
- solve a quadratic equation by completing the square
- interpret the solution of a quadratic equation graphically

1. Quadratic equations

Key Point 3

A **quadratic equation** is one which can be written in the form

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

where a, b and c are given numbers and x is the unknown whose value(s) must be found.

For example

$$2x^2 + 7x - 3 = 0, \qquad x^2 + x + 1 = 0, \qquad 0.5x^2 + 3x + 9 = 0$$

are all quadratic equations. To ensure the presence of the x^2 term, the number a, in the general expression $ax^2 + bx + c$ cannot be zero. However b or c may be zero, so that

$$4x^2 + 3x = 0, \quad 2x^2 - 3 = 0 \quad \text{and} \quad 6x^2 = 0$$

are also quadratic equations. Frequently, quadratic equations occur in non-standard form but where necessary they can be rearranged into standard form. For example

$$3x^2 + 5x = 8, \qquad \text{can be re-written as} \qquad 3x^2 + 5x - 8 = 0$$

$$2x^2 = 8x - 9, \qquad \text{can be re-written as} \qquad 2x^2 - 8x + 9 = 0$$

$$1 + x = \frac{1}{x}, \qquad \text{can be re-written as} \qquad x^2 + x - 1 = 0$$

To solve a quadratic equation we must find values of the unknown x which make the left-hand and right-hand sides equal. Such values are known as **solutions** or **roots** of the quadratic equation.

Note the difference between solving quadratic equations in comparison to solving linear equations. A quadratic equation will generally have **two** values of x (solutions) which satisfy it whereas a linear equation only has **one** solution.
We shall now describe three techniques for solving quadratic equations:

- factorisation
- completing the square
- using the quadratic formula

Exercises

1. Verify that $x = 2$ and $x = 3$ are both solutions of $x^2 - 5x + 6 = 0$.
2. Verify that $x = -2$ and $x = -3$ are both solutions of $x^2 + 5x + 6 = 0$.

2. Solution by factorisation

It may be possible to solve a quadratic equation by factorisation using the method described for factorising quadratic expressions in HELM 1.5, although you should be aware that not all quadratic equations can be easily factorised.

Example 10

Solve the equation $x^2 + 5x = 0$.

Solution

Factorising and equating each factor to zero we find

$$x^2 + 5x = 0 \quad \text{is equivalent to} \quad x(x+5) = 0$$

so that $x = 0$ and $x = -5$ are the two solutions.

Example 11

Solve the quadratic equation $x^2 + x - 6 = 0$.

Solution

Factorising the left hand side we find $x^2 + x - 6 = (x+3)(x-2)$ so that

$$x^2 + x - 6 = 0 \quad \text{is equivalent to} \quad (x+3)(x-2) = 0$$

When the product of two quantities equals zero, at least one of the two must equal zero. In this case either $(x+3)$ is zero or $(x-2)$ is zero. It follows that

$$x + 3 = 0, \qquad \text{giving} \qquad x = -3 \quad \text{or} \quad x - 2 = 0, \qquad \text{giving} \qquad x = 2$$

Here there are two solutions, $x = -3$ and $x = 2$.

These solutions can be checked quite easily by substitution back into the given equation.

Example 12

Solve the quadratic equation $2x^2 - 7x - 4 = 0$ by factorising the left-hand side.

Solution

Factorising the left hand side: $2x^2 - 7x - 4 = (2x+1)(x-4)$ so $2x^2 - 7x - 4 = 0$ is equivalent to $(2x+1)(x-4) = 0$. In this case either $(2x+1)$ is zero or $(x-4)$ is zero. It follows that $2x+1 = 0$, giving $x = -\frac{1}{2}$ or $x - 4 = 0$, giving $x = 4$

There are two solutions, $x = -\frac{1}{2}$ and $x = 4$.

Example 13

Solve the equation $4x^2 + 12x + 9 = 0$.

Solution

Factorising we find $4x^2 + 12x + 9 = (2x+3)(2x+3) = (2x+3)^2$

This time the factor $(2x+3)$ occurs twice. The original equation $4x^2 + 12x + 9 = 0$ becomes

$$(2x+3)^2 = 0 \quad \text{so that} \quad 2x + 3 = 0$$

and we obtain the solution $x = -\frac{3}{2}$. Because the factor $2x+3$ appears twice in the equation $(2x+3)^2 = 0$ we say that this root is a **repeated solution** or **double root**.

Task

Solve the quadratic equation $7x^2 - 20x - 3 = 0$.

First factorise the left-hand side:

Your solution

$7x^2 - 20x - 3 =$

Answer

$(7x+1)(x-3)$

Equate each factor is then equated to zero to obtain the two solutions:

Your solution

Solution 1: $x =$ Solution 2: $x =$

Answer

$-\frac{1}{7}$ and 3

Exercises

Solve the following equations by factorisation:

1. $x^2 - 3x + 2 = 0$	2. $x^2 - x - 2 = 0$	3. $x^2 + x - 2 = 0$
4. $x^2 + 3x + 2 = 0$	5. $x^2 + 8x + 7 = 0$	6. $x^2 - 7x + 12 = 0$
7. $x^2 - x - 20 = 0$	8. $4x^2 - 4 = 0$	9. $-x^2 + 2x - 1 = 0$
10. $3x^2 + 6x + 3 = 0$	11. $x^2 + 11x = 0$	12. $2x^2 + 2x = 0$

Answers The factors are found to be:

1. $1, 2$	2. $-1, 2$	3. $-2, 1$	4. $-1, -2$	5. $-7, -1$
6. $4, 3$	7. $-4, 5$	8. $1, -1$	9. 1 twice	10. -1 twice
11. $-11, 0$	12. $0, -1$			

3. Completing the square

The technique known as completing the square can be used to solve quadratic equations although it is applicable in many other circumstances too so it is well worth studying.

Example 14

(a) Show that $(x + 3)^2 = x^2 + 6x + 9$

(b) Hence show that $x^2 + 6x$ can be written as $(x + 3)^2 - 9$.

Solution

(a) Removing the brackets we find

$$(x + 3)^2 = (x + 3)(x + 3) = x^2 + 3x + 3x + 9 = x^2 + 6x + 9$$

(b) By subtracting 9 from both sides of the previous equation it follows that

$$(x + 3)^2 - 9 = x^2 + 6x$$

Example 15

(a) Show that $(x-4)^2 = x^2 - 8x + 16$

(b) Hence show that $x^2 - 8x$ can be written as $(x-4)^2 - 16$.

Solution

(a) Removing the brackets we find

$$(x-4)^2 = (x-4)(x-4) = x^2 - 4x - 4x + 16 = x^2 - 8x + 16$$

(b) Subtracting 16 from both sides we can write

$$(x-4)^2 - 16 = x^2 - 8x$$

We shall now generalise the results of Examples 14 and 15. Noting that

$$(x+k)^2 = x^2 + 2kx + k^2 \qquad \text{we can write} \qquad x^2 + 2kx = (x+k)^2 - k^2$$

Note that the constant term in the brackets on the right-hand side is always half the coefficient of x on the left. This process is called **completing the square**.

Key Point 4

Completing the Square

The expression $\quad x^2 + 2kx \quad$ is equivalent to $\quad (x+k)^2 - k^2$

Example 16

Complete the square for the expression $x^2 + 16x$.

Solution

Comparing $x^2 + 16x$ with the general form $x^2 + 2kx$ we see that $k = 8$. Hence

$$x^2 + 16x = (x+8)^2 - 8^2 = (x+8)^2 - 64$$

Note that the constant term in the brackets on the right, that is 8, is half the coefficient of x on the left, which is 16.

Example 17

Complete the square for the expression $5x^2 + 4x$.

Solution

Consider $5x^2 + 4x$. First of all the coefficient 5 is removed outside a bracket as follows

$$5x^2 + 4x = 5(x^2 + \frac{4}{5}x)$$

We can now complete the square for the quadratic expression in the brackets:

$$x^2 + \frac{4}{5}x = (x + \frac{2}{5})^2 - \left(\frac{2}{5}\right)^2 = (x + \frac{2}{5})^2 - \frac{4}{25}$$

Finally, multiplying both sides by 5 we find

$$5x^2 + 4x = 5\left((x + \frac{2}{5})^2 - \frac{4}{25}\right)$$

Completing the square can be used to solve quadratic equations as shown in the following Examples.

Example 18

Solve the equation $x^2 + 6x + 2 = 0$ by completing the square.

Solution

First of all just consider $x^2 + 6x$, and note that we can write this as

$$x^2 + 6x = (x + 3)^2 - 9$$

Then the quadratic equation can be written as

$$x^2 + 6x + 2 = (x + 3)^2 - 9 + 2 = 0 \quad \text{that is} \qquad (x + 3)^2 = 7$$

Taking the square root of both sides gives

$$x + 3 = \pm\sqrt{7} \quad \text{so} \quad x = -3 \pm \sqrt{7}$$

The two solutions are $x = -3 + \sqrt{7} = -0.3542$ and $x = -3 - \sqrt{7} = -5.6458$, to 4 d.p.

Example 19

Solve the equation $x^2 - 8x + 5 = 0$

Solution

First consider $x^2 - 8x$ which we can write as $x^2 - 8x = (x-4)^2 - 16$ so that the equation becomes

$$x^2 - 8x + 5 = (x-4)^2 - 16 + 5 = 0$$

$$\text{i.e.} \qquad \begin{aligned} (x-4)^2 &= 11 \\ x - 4 &= \pm\sqrt{11} \\ x &= 4 \pm \sqrt{11} \end{aligned}$$

So $x = 7.3166$ or $x = 0.6834$ (to 4 d.p.)

Task

Solve the equation $x^2 - 4x + 1 = 0$ by completing the square.

First examine the two left-most terms in the equation: $x^2 - 4x$. Complete the square for these terms:

Your solution

$x^2 - 4x =$

Answer

$(x-2)^2 - 4$

Use the above result to rewrite the equation $x^2 - 4x + 1 = 0$ in the form $(x-\ ?)^2 +\ ? = 0$:

Your solution

$x^2 - 4x + 1 =$

Answer

$(x-2)^2 - 4 + 1 = (x-2)^2 - 3 = 0$

From this now obtain the roots:

Your solution

Answer

$(x-2)^2 = 3$, so $x - 2 = \pm\sqrt{3}$. Therefore $x = 2 \pm \sqrt{3}$ so $x = 3.7321$ or 0.2679 to 4 d.p.

Exercises

1. Solve the following quadratic equations by completing the square.

   (a) $x^2 - 3x = 0$

   (b) $x^2 + 9x = 0.$

   (c) $2x^2 - 5x + 2 = 0$

   (d) $6x^2 - x - 1 = 0$

   (e) $-5x^2 + 6x - 1 = 0$

   (f) $-x^2 + 4x - 3 = 0$

2. A chemical manufacturer found that the sales figures for a certain chemical X_2O depended on its selling price. At present, the company can sell all of its weekly production of 300 t at a price of £600 / t. The company's market research department advised that the amount sold would decrease by only 1 t per week for every £2 / t increase in the price of X_2O. If the total production costs are made up of a fixed cost of £30000 per week, plus £400 per t of product, show that the weekly profit is given by

$$P = -\frac{x^2}{2} + 800x - 270000$$

   where x is the new price per t of X_2O. Complete the square for the above expression and hence find

   (a) the price which maximises the weekly profit on sales of X_2O

   (b) the maximum weekly profit

   (c) the weekly production rate

Answers

1. (a) $0, 3$ (b) $0, -9$ (c) $2, \frac{1}{2}$ (d) $\frac{1}{2}, -\frac{1}{3}$ (e) $\frac{1}{5}, 1$ (f) $1, 3$

2. (a) £800 / t, (b) £50000 /wk, (c) 200 t / wk

4. Solution by formula

When it is difficult to factorise a quadratic equation, it may be possible to solve it using a formula which is used to calculate the roots. The formula is obtained by completing the square in the general quadratic $ax^2 + bx + c$. We proceed by removing the coefficient of a:

$$ax^2 + bx + c = a\{x^2 + \frac{b}{a}x + \frac{c}{a}\} = a\{(x + \frac{b}{2a})^2 + \frac{c}{a} - \frac{b^2}{4a^2}\}$$

Thus the solution of $ax^2 + bx + c = 0$ is the same as the solution to

$$(x + \frac{b}{2a})^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

So, solving: $(x+\frac{b}{2a})^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$ which leads to $x = -\frac{b}{2a} \pm \sqrt{-\frac{c}{a} + \frac{b^2}{4a^2}}$

Simplifying this expression further we obtain the important result:

Key Point 5

Quadratic Formula

If $ax^2 + bx + c = 0, \quad a \neq 0$ then the two solutions (roots) are

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

To apply the formula to a specific quadratic equation it is necessary to identify carefully the values of a, b and c, paying particular attention to the signs of these numbers. Substitution of these values into the formula then gives the desired solutions.

Note that if the quantity $b^2 - 4ac$ (called the **discriminant**) is a positive number we can take its square root and the formula will produce two values known as **distinct real roots**. If $b^2 - 4ac = 0$ there will be one value only known as a **repeated root** or **double root**. The value of this root is $x = -\frac{b}{2a}$. Finally if $b^2 - 4ac$ is negative we say the equation possesses **complex roots**. These require special treatment and are described in HELM 10.

Key Point 6

When finding roots of the quadratic equation $ax^2 + bx + c = 0$ first calculate the discrinimant

$$b^2 - 4ac$$

- If $b^2 - 4ac > 0$ the quadratic has two real distinct roots
- If $b^2 - 4ac = 0$ the quadratic has two real and equal roots
- If $b^2 - 4ac < 0$ the quadratic has no real roots: there are two complex roots

Example 20

Compare each given equation with the standard form $ax^2+bx+c=0$ and identify a, b and c. Calculate b^2-4ac in each case and use this information to state the nature of the roots.

(a) $3x^2+2x-7=0$ (b) $3x^2+2x+7=0$

(c) $3x^2-2x+7=0$ (d) $x^2+x+2=0$

(e) $-x^2+3x-\frac{1}{2}=0$ (f) $5x^2-3=0$

(g) $x^2-2x+1=0$ (h) $2p^2-4p=0$

(i) $-p^2+4p-4=0$

Solution

(a) $a=3,\ b=2,\ c=-7$. So $b^2-4ac=(2)^2-4(3)(-7)=88$.
The roots are real and distinct.

(b) $a=3,\ b=2, c=7$. So $b^2-4ac=(2)^2-4(3)(7)=-80$.
The roots are complex.

(c) $a=3,\ b=-2,\ c=7$. So $b^2-4ac=(-2)^2-4(3)(7)=-80$.
The roots are complex.

(d) $a=1,\ b=1,\ c=2$. So $b^2-4ac=1^2-4(1)(2)=-7$.
The roots are complex.

(e) $a=-1,\ b=3,\ c=-\frac{1}{2}$. So $b^2-4ac=3^2-4(-1)(-\frac{1}{2})=7$.
The roots are real and distinct.

(f) $a=5,\ b=0,\ c=-3$. So $b^2-4ac=0-4(5)(-3)=60$.
The roots are real and distinct.

(g) $a=1,\ b=-2,\ c=1$. So $b^2-4ac=(-2)^2-4(1)(1)=0$.
The roots are real and equal.

(h) $a=2, b=-4, c=0$. So $b^2-4ac=(-4)^2-4(2)(0)=16$
The roots are real and distinct.

(i) $a=-1, b=4, c=-4$. So $b^2-4ac=(-4)^2-4(-1)(-4)=0$
The roots are real and equal.

Example 21

Solve the quadratic equation $2x^2 + 3x - 6 = 0$ using the formula.

Solution

We compare the given equation with the standard form $ax^2 + bx + c = 0$ in order to identify a, b and c. We see that here $a = 2$, $b = 3$ and $c = -6$. Note particularly the sign of c. Substituting these values into the formula we find

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4(2)(-6)}}{(2)(2)} = \frac{-3 \pm \sqrt{9 + 48}}{4} = \frac{-3 \pm 7.5498}{4}$$

Hence, to 4 d.p., the two roots are $x = 1.1375$, if the positive sign is taken and $x = -2.6375$ if the negative sign is taken. However, it is often sufficient to leave the solution in the so-called **surd form** $x = \dfrac{-3 \pm \sqrt{57}}{4}$, which is exact.

Task

Solve the equation $3x^2 - x - 6 = 0$ using the quadratic formula.

First identify a, b and c:

Your solution

$a =$ $\qquad$ $b =$ $\qquad$ $c =$

Answer

$a = 3, \qquad b = -1, \qquad c = -6$

Substitute these values into the formula and simplify:

Your solution

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\qquad$ so $\quad x =$

Answer

$$\frac{-(-1) \pm \sqrt{(-1)^2 - (4)(3)(-6)}}{(2)(3)} = \frac{1 \pm \sqrt{73}}{6}$$

Finally, calculate the values of x to 4 d.p.:

Your solution

$x =$ $\qquad$ or $\qquad$ $x =$

Answer

$1.5907, -1.2573$

Engineering Example 1

Undersea cable fault location

Introduction

The voltage (V), current (I) and resistance (R) in an electrical circuit are related by Ohm's law i.e. $V = IR$. If there are two resistances (R_1 and R_2) in an electrical circuit, they may be in series, in which case the total resistance (R) is given by $R = R_1 + R_2$. Or they may be in parallel in which case the total resistance is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

In 1871 the telephone cable between England (A) and Denmark (B) developed a fault, due to a short circuit under the sea (see Figure 2). Oliver Heaviside, an electrical engineer, came up with a very simple method to find the location of the fault. He assumed that the cable had a uniform resistance per unit length. Heaviside performed two tests:

(1) connecting a battery (voltage E) at A, with the circuit open at B, he measured the resulting current I_1,

(2) connecting the same battery at A, with the cable earthed at B, he measured the current I_2.

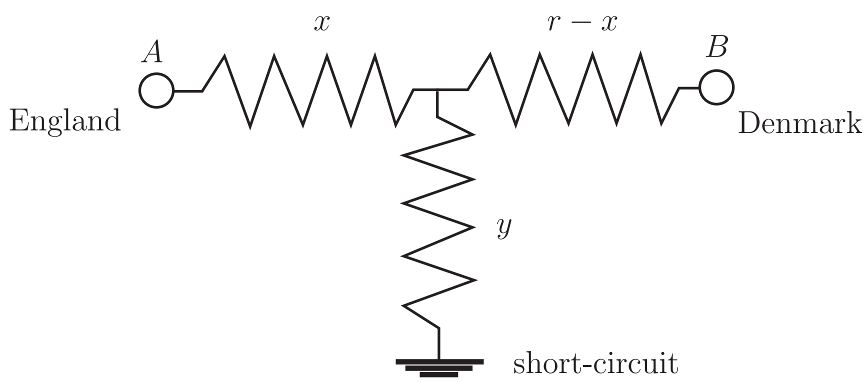


Figure 2: Schematic of the undersea cable

In the first measurement the resistances up to the cable fault and between the fault and the short circuit are in series and in the second experiment the resistances beyond the fault and between the fault and the short circuit are in parallel.

Problem in words

Use the information from the measurements to deduce the location of the fault.

Mathematical statement of problem

(a) Denote the resistances of the various branches by the symbols shown in Figure 2.

(b) Use Ohm's law to write down expressions that apply to each of the two measurements.

(c) Eliminate y from these expressions to obtain an expression for x.

Mathematical analysis

(a) In the first experiment the total circuit resistance is $x + y$. In the second experiment, the total circuit resistance is given by:

$$x + \left(\frac{1}{r-x} + \frac{1}{y}\right)^{-1}$$

So application of Ohm's law to each experimental situation gives:

$$E = I_1(x + y) \qquad (1)$$

$$E = I_2(x + \left(\frac{1}{r-x} + \frac{1}{y}\right)^{-1}) \qquad (2)$$

Rearrange Equation (1) to give $\dfrac{E}{I_1} - x = y$

Substitute for y in Equation (2), divide both sides by I_2 and introduce $\dfrac{E}{I_1} = r_1$ and $\dfrac{E}{I_2} = r_2$:

$$r_2 = (x + \left(\frac{1}{r-x} + \frac{1}{y}\right)^{-1})$$

Use a common denominator for the fractions on the right-hand side:

$$r_2 = (x + \left(\frac{(r-x)(r_1-x)}{r_1 - x + r - x}\right)) = \frac{x(r_1 + r - 2x) + (r-x)(r_1-x)}{(r_1 + r - 2x)}$$

Multiply through by $(r_1 + r - 2x)$:

$$r_2(r_1 + r - 2x) = x(r_1 + r - 2x) + (r-x)(r_1 - x)$$

Rearrange as a quadratic for x:

$$x^2 - 2r_2x - rr_1 + r_2r_1 + rr_2 = 0$$

Use the standard formula for solving quadratic equations

with $a = 1, b = -2r_2$ and $c = -rr_1 + r_2r_1 + rr_2$:

$$x = \frac{2r_2 \pm \sqrt{4r_2^2 - 4(-rr_1 + r_2r_1 + rr_2)}}{2} = r_2 \pm \sqrt{(r - r_2)(r_1 - r_2)}$$

Only positive solutions would be of interest.

Engineering Example 2

Estimating the mass of a pipe

Introduction

Sometimes engineers have to estimate component weights from dimensions and material properties. On some occasions, engineers prefer use of approximate formulae to exact ones as long as they are sufficiently accurate for the purpose. This Example introduces both of these aspects.

Problem in words

(a) Find the mass of a given length of pipe in terms of its inner and outer diameters and the density of the pipe material.

(b) Find the wall thickness of the pipe if the inner diameter is 0.15 m, the density is 7900 kg m^{-3} and the mass per unit length of pipe is 40 kg m^{-1}.

(c) Find an approximate method for calculating the mass of a given length of a thin-walled pipe and calculate the maximum ratio of inner and outer diameters that give an error of less than 10% when using the approximate method.

Mathematical statement of problem

(a) Denote the length of the pipe by L m and inside and outside diameters by d_i m and d_o m, respectively and the density by ρ kg m^{-3}. Assume that the pipe is cylindrical so its cross section corresponds to the gap between concentric circles (this is called an annulus or annular region - see HELM 2.6). Calculate the difference in cross sectional areas by using the formula for the area of a circle ($A = \pi r^2$ where r is the radius) and multiply by the density and length to obtain mass (m).

(b) Rearrange the equation in terms of wall thickness (d m) and inner diameter. Substitute the given values to determine the wall thickness.

(c) Approximate the resulting expression for small values of $(d_o - d_i)$. Calculate the percentage difference in predictions between the original and approximate formulae for various numerical values of d_i/d_o.

Mathematical analysis

(a) The cross section of a cylindrical pipe is a circular annulus. The area of a circle is given by $\pi r^2 = \dfrac{\pi}{4}d^2$, since $r = d/2$ if d is the diameter. So the area of the outer circle is $\dfrac{\pi}{4}d_o^2$ and that of the inner circle is $\dfrac{\pi}{4}d_i^2$. This means that the mass m kg of length L m of the pipe is given by

$$m = \frac{\pi}{4}(d_0^2 - d_i^2)L\rho$$

(b) Denote the pipe wall thickness by δ so $d_o = d_i + 25$.

Use $(d_o^2 - d_i^2) = (d_o - d_i)(d_o + d_i) = 2\delta(2d_i + 2\delta)$. So $\quad m = \pi\delta(d_i + \delta)L\rho$

Given that $m/L = 40,\ d_i = 0.15$ and $r = 7900$,

$$40 = pd(0.15 + d)7900.$$

Rearrange this equation as a quadratic in δ,

$$\delta^2 + 0.15\delta - 4\pi/790 = 0$$

Solve this quadratic using the standard formula with $a = 1, b = 0.15$ and $c = 4\pi/790$. Retain only the positive solution to give $\delta = 0.072$, i.e. the pipe wall thickness is 72 mm.

(c) If δ is small then $(d_o - d_i)$ is small and $d_i + \delta \approx d_i$. So the expression for m in terms of δ may be written

$$m \approx \pi\delta d_i L\rho$$

The graph in Figure 3 shows that the percentage error from using the approximate formula for the mass of the pipe exceeds 10% only if the inner diameter is less than 82% of the outer diameter.

The percentage error from using the approximate formula can be calculated from (exact result $-$ approximate result)/(exact result) $\times$ 100% for various values of the ratio of inner to outer diameters. In the graph the error is plotted for diameter ratios between 0.75 and 1.

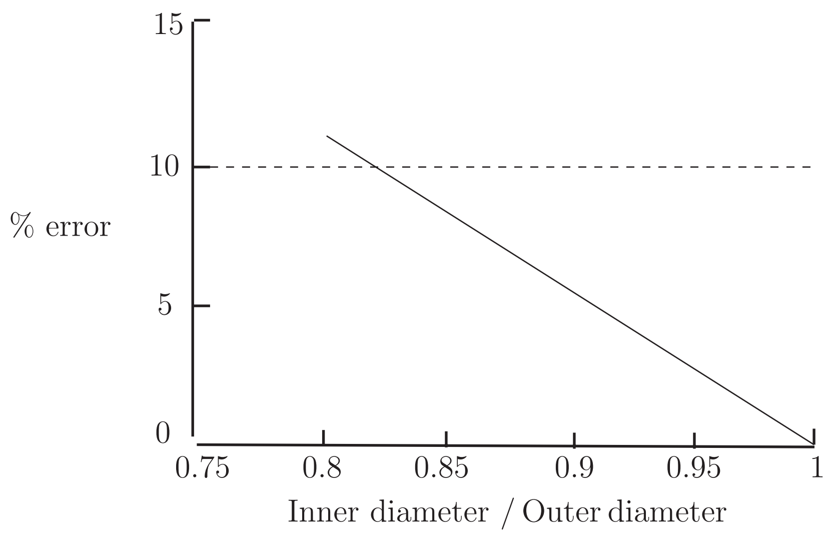


Figure 3

Comment

The graph shows also that the error is 1% or less for diameter ratios > 0.98.

Exercises

Solve the following quadratic equations by using the formula. Give answers exactly (where possible) or to 4 d.p.:

1. $x^2 + 8x + 1 = 0$	2. $x^2 + 7x - 2 = 0$	3. $x^2 + 6x - 2 = 0$
4. $-x^2 + 3x + 1 = 0$	5. $-2x^2 - 3x + 1 = 0$	6. $2x^2 + 5x - 3 = 0$

Answers

1. -0.1270, -7.8730	2. $-7.2749, 0.2749$	3. 0.3166, -6.3166
4. 3.3028, -0.3028	5. $-1.7808, 0.2808$	6. $\frac{1}{2}, -3$

5. Geometrical representation of quadratics

We can plot a graph of the function $y = ax^2 + bx + c$ (given the values of a, b and c). If the graph crosses the horizontal axis it will do so when $y = 0$, and so the x coordinates at such points are solutions of $ax^2 + bx + c = 0$. Depending on the sign of a and of the nature of the solutions there are essentially six different types of graph that can occur. These are displayed in Figure 4.

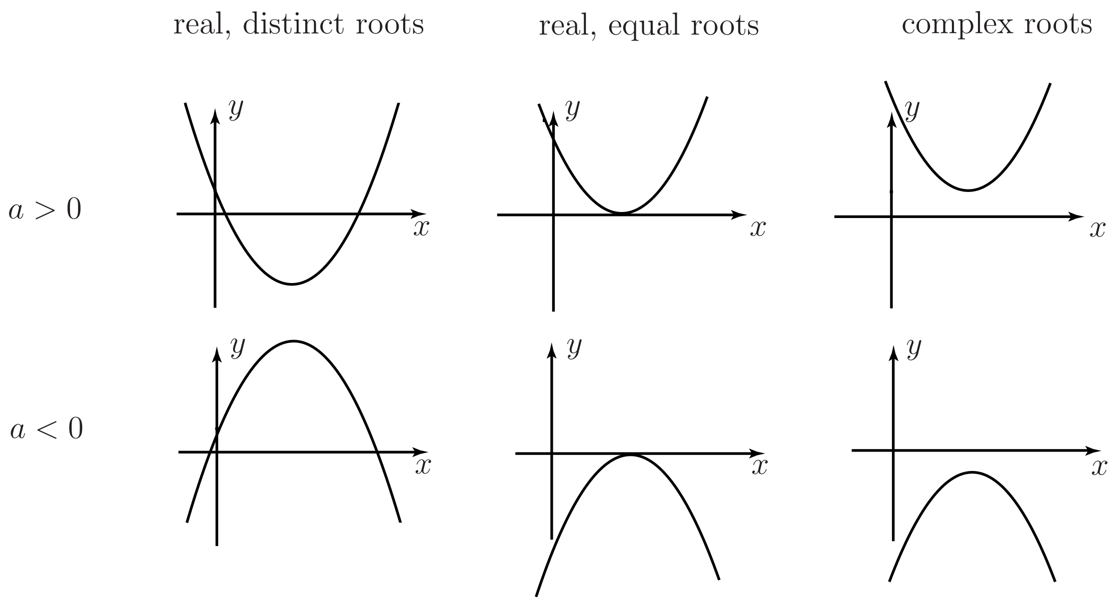


Figure 4: The possible graphs of a quadratic $y = ax^2 + bx + c$

Sometimes a graph of the quadratic is used to locate the solutions; however, this approach is generally inaccurate. This is illustrated in the following example.

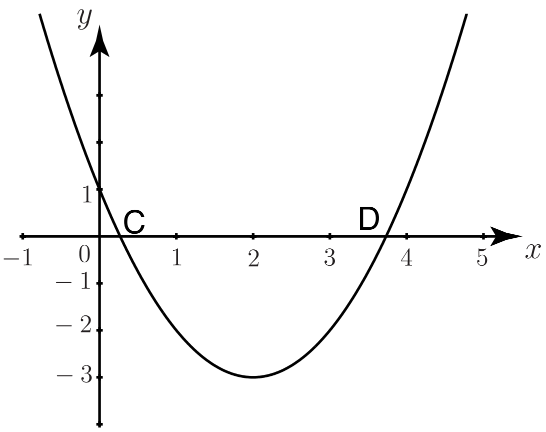
Example 22

Solve the equation $x^2 - 4x + 1 = 0$ by plotting a graph of the function:

$$y = x^2 - 4x + 1$$

Solution

By constructing a table of function values we can plot the graph as shown in Figure 5.



x	0	1	2	3	4
y	1	−2	−3	−2	1

Figure 5: The graph of $y = x^2 - 4x + 1$ cuts the x axis at C and D

The solutions of the equation $x^2 - 4x + 1 = 0$ are found by looking for points where the graph crosses the horizontal axis. The two points are approximately $x = 0.3$ and $x = 3.7$ marked C and D on the Figure.

Exercises

1. Solve the following quadratic equations giving exact numeric solutions. Use whichever method you prefer

   (a) $x^2 - 9 = 0$ (b) $s^2 - 25 = 0$
   (c) $3x^2 - 12 = 0$ (d) $x^2 - 5x + 6 = 0$
   (e) $6s^2 + s - 15 = 0$ (f) $p^2 + 7p = 0$

2. Solve the equation $2x^2 - 3x - 7 = 0$ giving solutions rounded to 4 d.p.

3. Solve the equation $2t^2 + 3t - 4$ giving the solutions in surd form.

Answers

1(a) $x = 3, -3$, (b) $s = 5, -5$, (c) $x = 2, -2$, (d) $x = 3, 2$, (e) $s = 3/2, -5/3$,

(f) $p = 0, -7$. 2. $-2.7656, 1.2656$. 3. $\dfrac{-3 \pm \sqrt{43}}{4}$

Solving Polynomial Equations

3.3

Introduction

Linear and quadratic equations, dealt within Sections 3.1 and 3.2, are members of a class of equations, called **polynomial equations**. These have the general form:

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 = 0$$

in which x is a variable and $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are given constants. Also n must be a positive integer and $a_n \neq 0$. Examples include $x^3 + 7x^2 + 3x - 2 = 0$, $\quad 5x^4 - 7x^2 = 0$ and $-x^6 + x^5 - x^4 = 0$. In this Section you will learn how to factorise some polynomial expressions and solve some polynomial equations.

Prerequisites

Before starting this Section you should . . .

- be able to solve linear and quadratic equations

Learning Outcomes

On completion you should be able to . . .

- recognise and solve some polynomial equations

1. Multiplying polynomials together

Key Point 7

A **polynomial expression** is one of the form

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where a_0, a_1, ..., a_n are known coefficients (numbers), $a_n \neq 0$, and x is a variable.

n must be a positive integer.

For example $x^3 - 17x^2 + 54x - 8$ is a polynomial expression in x. The polynomial may be expressed in terms of a variable other than x. So, the following are also polynomial expressions:

$$t^3 - t^2 + t - 3 \qquad z^5 - 1 \qquad w^4 + 10w^2 - 12 \qquad s + 1$$

Note that only non-negative whole number powers of the variable (usually x) are allowed in a polynomial expression. In this Section you will learn how to factorise simple polynomial expressions and how to solve some polynomial equations. You will also learn the technique of **equating coefficients**. This process is very important when we need to perform calculations involving partial fractions which will be considered in Section 6.

The **degree** of a polynomial is the highest power to which the variable is raised. Thus $x^3 + 6x + 2$ has degree 3, $t^6 - 6t^4 + 2t$ has degree 6, and $5x + 2$ has degree 1.

Let us consider what happens when two polynomials are multiplied together. For example

$$(x + 1)(3x - 2)$$

is the product of two first degree polynomials. Expanding the brackets we obtain

$$(x + 1)(3x - 2) = 3x^2 + x - 2$$

which is a second degree polynomial.

In general we can regard a second degree polynomial, or quadratic, as the product of two first degree polynomials, provided that the quadratic can be factorised. Similarly

$$(x - 1)(x^2 + 3x - 7) = x^3 + 2x^2 - 10x + 7$$

is a third degree, or **cubic**, polynomial which is thus the product of a linear polynomial and a quadratic polynomial.

In general we can regard a cubic polynomial as the product of a linear polynomial and a quadratic polynomial or the product of three linear polynomials. This fact will be important in the following Section when we come to factorise cubics.

Key Point 8

A cubic expression can always be formulated as a linear expression times a quadratic expression.

Task

If $x^3 - 17x^2 + 54x - 8 = (x - 4) \times$ (a polynomial), state the degree of the undefined polynomial.

Your solution

Answer

second.

Task

(a) If $3x^2 + 13x + 4 = (x + 4) \times$ (a polynomial), state the degree of the undefined polynomial.

(b) What is the coefficient of x in this unknown polynomial ?

Your solution

(a) (b)

Answer

(a) First. (b) It must be 3 in order to generate the term $3x^2$ when the brackets are removed.

Task

If $2x^2 + 5x + 2 = (x + 2)\times$ (a polynomial), what must be the coefficient of x in this unknown polynomial ?

Your solution

Answer

It must be 2 in order to generate the term $2x^2$ when the brackets are removed.

Two quadratic polynomials are multiplied together. What is the degree of the resulting polynomial?

Your solution

Answer
Fourth degree.

2. Factorising polynomials and equating coefficients

We will consider how we might find the solution to some simple polynomial equations. An important part of this process is being able to express a complicated polynomial into a product of simpler polynomials. This involves **factorisation**.

Factorisation of polynomial expressions can be achieved more easily if one or more of the factors is already known. This requires a knowledge of the technique of 'equating coefficients' which is illustrated in the following example.

Example 23

Factorise the expression $x^3 - 17x^2 + 54x - 8$ given that one of the factors is $(x - 4)$.

Solution

Given that $x - 4$ is a factor we can write

$$x^3 - 17x^2 + 54x - 8 = (x - 4) \times (\text{a quadratic polynomial})$$

The polynomial must be quadratic because the expression on the left is cubic and $x - 4$ is linear. Suppose we write this quadratic as $ax^2 + bx + c$ where a, b and c are unknown numbers which we need to find. Then

$$x^3 - 17x^2 + 54x - 8 = (x - 4)(ax^2 + bx + c)$$

Removing the brackets on the right and collecting like terms together we have

$$x^3 - 17x^2 + 54x - 8 = ax^3 + (b - 4a)x^2 + (c - 4b)x - 4c$$

Solution (contd.)

Like terms are those which involve the same power of the variable (x).

Equating coefficients means that we compare the coefficients of each term on the left with the corresponding term on the right. Thus if we look at the x^3 terms on each side we see that $x^3 = ax^3$ which implies a must equal 1. Similarly by equating coefficients of x^2 we find $-17 = b - 4a$ With $a = 1$ we have $-17 = b - 4$ so b must equal -13. Finally, equating constant terms we find $-8 = -4c$ so that $c = 2$.

As a check we look at the coefficient of x to ensure it is the same on both sides. Now that we know $a = 1, b = -13, c = 2$ we can write the polynomial expression as

$$x^3 - 17x^2 + 54x - 8 = (x - 4)(x^2 - 13x + 2)$$

Exercises

Factorise into a quadratic and linear product the given polynomial expressions

1. $x^3 - 6x^2 + 11x - 6$, given that $x - 1$ is a factor
2. $x^3 - 7x - 6$, given that $x + 2$ is a factor
3. $2x^3 + 7x^2 + 7x + 2$, given that $x + 1$ is a factor
4. $3x^3 + 7x^2 - 22x - 8$, given that $x + 4$ is a factor

Answers

1. $(x - 1)(x^2 - 5x + 6)$, 2. $(x + 2)(x^2 - 2x - 3)$, 3. $(x + 1)(2x^2 + 5x + 2)$,
4. $(x + 4)(3x^2 - 5x - 2)$.

3. Polynomial equations

When a polynomial expression is equated to zero, a polynomial equation is obtained. Linear and quadratic equations, which you have already met, are particular types of polynomial equation.

Key Point 9

A **polynomial equation** has the form

$$a_nx^n + a_{n-1}x^{n-1} + \dots a_2x^2 + a_1x + a_0 = 0$$

where a_0, a_1, ..., a_n are known coefficients, $a_n \neq 0$, and x represents an unknown whose value(s) are to be found.

Polynomial equations of low degree have special names. A polynomial equation of degree 1 is a linear equation and such equations have been solved in Section 3.1. Degree 2 polynomials are called quadratics; degree 3 polynomials are called cubics; degree 4 equations are called quartics and so on. The following are examples of polynomial equations:

$$5x^6 - 3x^4 + x^2 + 7 = 0, \qquad -7x^4 + x^2 + 9 = 0, \qquad t^3 - t + 5 = 0, \qquad w^7 - 3w - 1 = 0$$

Recall that the **degree** of the equation is the highest power of x occurring. The **solutions** or **roots** of the equation are those values of x which satisfy the equation.

Key Point 10

A polynomial equation of degree n has n roots.

Some (possibly all) of the roots may be repeated.

Some (possibly all) of the roots may be complex.

Example 24

Verify that $x = -1$, $x = 1$ and $x = 0$ are solutions (roots) of the equation

$$x^3 - x = 0$$

Solution

We substitute each value in turn into $x^3 - x$.

$$(-1)^3 - (-1) = -1 + 1 = 0$$

so $x = -1$ is clearly a root.

It is easy to verify similarly that $x = 1$ and $x = 0$ are also solutions.

In the next subsection we will consider ways in which polynomial equations of higher degree than quadratic can be solved.

Exercises

Verify that the given values are solutions of the given equations.

1. $x^2 - 5x + 6 = 0, \qquad x = 3,\ x = 2$
2. $2t^3 + t^2 - t = 0, \qquad t = 0,\ t = -1,\ t = \frac{1}{2}.$

4. Solving polynomial equations when one solution is known

In Section 3.2 we gave a formula which can be used to solve quadratic equations. Unfortunately when dealing with equations of higher degree no simple formulae exist. If one of the roots can be spotted or is known we can sometimes find the others by the method shown in the next Example.

Example 25

Let the polynomial expression $x^3 - 17x^2 + 54x - 18$ be denoted by $P(x)$. Verify that $x = 4$ is a solution of the equation $P(x) = 0$. Hence find the other solutions.

Solution

We substitute $x = 4$ into the polynomial expression $P(x)$:

$$P(4) = 4^3 - 17(4^2) + 54(4) - 8 = 64 - 272 + 216 - 8 = 0$$

So, when $x = 4$ the left-hand side equals zero. Hence $x = 4$ is indeed a solution. Knowing that $x = 4$ is a root we can state that $(x-4)$ must be a factor of $P(x)$. Therefore $P(x)$ can be re-written as a product of a linear and a quadratic term:

$$P(x) = x^3 - 17x^2 + 54x - 8 = (x - 4) \times \text{ (quadratic polynomial)}$$

The quadratic polynomial has already been found in a previous task so we deduce that the given equation can be written

$$P(x) = x^3 - 17x^2 + 54x - 8 = (x - 4)(x^2 - 13x + 2) = 0$$

In this form we see that $\quad x - 4 = 0 \quad$ or $\quad x^2 - 13x + 2 = 0$

The first equation gives $x = 4$ which we already knew.

The second equation must be solved using one of the methods for solving quadratic equations given in Section 3.2. For example, using the formula we find

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{with } a = 1,\ b = -13,\ c = 2 \\ &= \frac{13 \pm \sqrt{(-13)^2 - 4.1.2}}{2} \\ &= \frac{13 \pm \sqrt{161}}{2} = \frac{13 \pm 12.6886}{2} \end{aligned}$$

So $x = 12.8443$ and $x = 0.1557$ are roots of $x^2 - 13x + 2$.

Hence the three solutions of $P(x) = 0$ are $x = 4$, $x = 12.8443$ and $x = 0.1557$, to 4 d.p.

Solve the equation $x^3 + 8x^2 + 16x + 3 = 0$ given that $x = -3$ is a root.
Consider the equation $x^3 + 8x^2 + 16x + 3 = 0$.

Given that $x = -3$ is a root state a linear factor of the cubic:

Your solution

Answer

$x + 3$

The cubic can therefore be expressed as

$$x^3 + 8x^2 + 16x + 3 = (x + 3)(ax^2 + bx + c)$$

where $a, b,$ and c are constants. These can be found by expanding the right-hand side.

Expand the right-hand side:

Your solution

Answer

$$x^3 + 8x^2 + 16x + 3 = ax^3 + (3a + b)x^2 + (3b + c)x + 3c$$

Equate coefficients of x^3 to find a:

Your solution

Answer

1

Equate constant terms to find c:

Your solution

Answer

$3 = 3c$ so that $c = 1$

Equate coefficients of x^2 to find b:

Your solution

Answer
$8 = 3a + b$ so $b = 5$

This enables us to write the equation as $(x+3)(x^2+5x+1) = 0$ so $x+3 = 0$ or $x^2+5x+1 = 0$. Now solve the quadratic and state all three roots:

Your solution

Answer
The quadratic equation can be solved using the formula to obtain $x = -4.7913$ and $x = -0.2087$. Thus the three roots of $x^3+8x^2+16x+3$ are $x = -3$, $x = -4.7913$ and $x = -0.2087$.

Exercises

1. Verify that the given value is a solution of the equation and hence find all solutions:

(a) $x^3+7x^2+11x+2 = 0, \qquad x = -2$ (b) $2x^3+11x^2-2x-35 = 0, \qquad x = -5$

2. Verify that $x = 1$ and $x = 2$ are solutions of $x^4+4x^3-17x^2+8x+4$ and hence find all solutions.

Answers

1(a) -2, -0.2087, -4.7913 1(b) $-5, -2.1375, 1.6375$
2. 1,2, -0.2984, -6.7016

5. Solving polynomial equations graphically

Polynomial equations, particularly of high degree, are difficult to solve unless they take a particularly simple form. A useful guide to the approximate values of the solutions can be obtained by sketching the polynomial, and discovering where the curve crosses the x-axis. The real roots of the polynomial equation $P(x) = 0$ are given by the values of the intercepts of the function $y = P(x)$ with the x-axis because on the x-axis $y = P(x)$, is zero. Computer software packages and graphics calculators exist which can be used for plotting graphs and hence for solving polynomial equations approximately. Suppose the graph of $y = P(x)$ is plotted and takes a form similar to that shown in Figure 6.

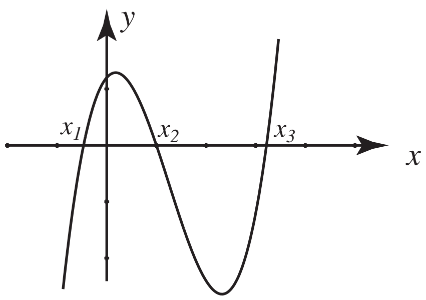


Figure 6: A polynomial function which cuts the x axis at points x_1, x_2 and x_3.

The graph intersects the x axis at $x = x_1$, $x = x_2$ and $x = x_3$ and so the equation $P(x) = 0$ has three roots x_1, x_2 and x_3, because $P(x_1) = 0$, $P(x_2) = 0$ and $P(x_3) = 0$.

Example 26

Plot a graph of the function $y = 4x^4 - 15x^2 + 5x + 6$ and hence approximately solve the equation $4x^4 - 15x^2 + 5x + 6 = 0$.

Solution

The graph has been plotted here with the aid of a computer graph plotting package and is shown in Figure 7. By hand, a less accurate result would be produced, of course.

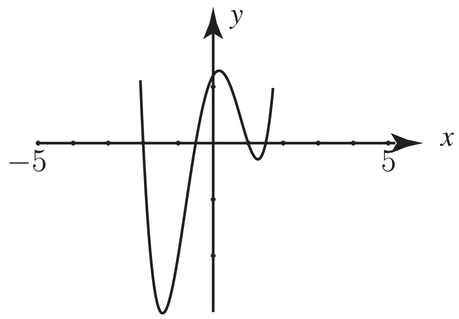


Figure 7: Graph of $y = 4x^4 - 15x^2 + 5x + 6$

The solutions of the equation are found by looking for where the graph crosses the horizontal axis. Careful examination shows the solutions are at or close to $x = 1$, $x = 1.5$, $x = -0.5$, $x = -2$.

An important feature of the graph of a polynomial is that it is **continuous**. There are never any gaps or jumps in the curve. Polynomial curves never turn back on themselves in the horizontal direction, (unlike a circle). By studying the graph in Figure 6 you will see that if we choose any two values of x, say a and b, such that $y(a)$ and $y(b)$ have opposite signs, then at least one root lies between $x = a$ and $x = b$.

Exercises

1. Factorise $x^3 - x^2 - 65x - 63$ given that $(x + 7)$ is a factor.
2. Show that $x = -1$ is a root of $x^3+11x^2+31x+21 = 0$ and locate the other roots algebraically.
3. Show that $x = 2$ is a root of $x^3 - 3x - 2 = 0$ and locate the other roots.
4. Solve the equation $x^4 - 2x^2 + 1 = 0$.
5. Factorise $x^4 - 7x^3 + 3x^2 + 31x + 20$ given that $(x + 1)$ is a factor.
6. Given that two of the roots of $x^4 + 3x^3 - 7x^2 - 27x - 18 = 0$ have the same modulus but different sign, solve the equation.

   (Hint - let two of the roots be α and $-\alpha$ and use the technique of equating coefficients).
7. Consider the polynomial $P(x) = 5x^3 - 47x^2 + 84x$. By evaluating $P(2)$ and $P(3)$ show that at least one root of $P(x) = 0$ lies between $x = 2$ and $x = 3$.
8. Without solving the equation or using a graphical calculator, show that $x^4 + 4x - 1 = 0$ has a root between $x = 0$ and $x = 1$.

Answers

1. $(x + 7)(x + 1)(x - 9)$
2. $x = -1, -3, -7$
3. $x = 2, -1$ (repeated)
4. $x = -1, 1$ (each root repeated)
5. $(x + 1)^2(x - 4)(x - 5)$
6. $(x + 3)(x - 3)(x + 1)(x + 2)$

Solving Simultaneous Linear Equations

3.4

Introduction

Equations often arise in which there is more than one unknown quantity. When this is the case there will usually be more than one equation involved. For example in the two linear equations

$$7x + y = 9, \quad -3x + 2y = 1$$

there are two unknowns: x and y. In order to solve the equations we must find values for x and y that satisfy both of the equations simultaneously. The two equations are called **simultaneous equations**. You should verify that the solution of these equations is $x = 1$, $y = 2$ because by substituting these values into both equations, the left-hand and right-hand sides are equal.

In this Section we shall show how two simultaneous equations can be solved either by a method known as **elimination** or by drawing graphs. In realistic problems which arise in mathematics and in engineering there may be many equations with many unknowns. Such problems cannot be solved using a graphical approach (we run out of dimensions in our 3-dimensional world!). Solving these more general problems requires the use of more general elimination procedures or the use of matrix algebra. Both of these topics are discussed in later Workbooks.

Prerequisites

Before starting this Section you should ...

- be able to solve linear equations

Learning Outcomes

On completion you should be able to ...

- solve pairs of simultaneous linear equations

1. Solving simultaneous equations by elimination

One way of solving simultaneous equations is by **elimination**. As the name implies, elimination, involves removing one or more of the unknowns. Note that if **both** sides of an equation are multiplied or divided by a non-zero number an exactly equivalent equation results. For example, if we are given the equation

$$x + 4y = 5$$

then by multiplying both sides by 7 we find

$$7x + 28y = 35$$

and this modified equation is equivalent to the original one.
Given two simultaneous equations, elimination of one unknown can be achieved by modifying the equations so that the coefficients of that unknown in each equation are the same and then subtracting one modified equation from the other. Consider the following example.

Example 27

Solve the simultaneous equations

$$3x + 5y = 31 \qquad (1)$$

$$2x + 3y = 20 \qquad (2)$$

Solution

We first try to modify each equation so that the coefficient of x is the same in both equations. This can be achieved if Equation (1) is multiplied by 2 and Equation (2) is multiplied by 3. This gives

$$\begin{array}{rcrcl} 6x & + & 10y & = & 62 \\ 6x & + & 9y & = & 60 \end{array}$$

Now the unknown x can be eliminated if the second equation is subtracted from the first:

$$\begin{array}{rrcrcr} & 6x & + & 10y & = & 62 \\ \text{subtract} & 6x & + & 9y & = & 60 \\ \hline & 0x & + & 1y & = & 2 \end{array}$$

The result implies that $1y = 2$ and we see immediately that y must equal 2. To find x we substitute the value found for y into either of the given Equations (1) or (2). For example, using Equation (1),

$$\begin{aligned} 3x + 5(2) &= 31 \\ 3x &= 21 \\ x &= 7 \end{aligned}$$

Thus the solution of the simultaneous equations is $x = 7$, $y = 2$.

N.B. You should always check your solution by substituting back into both of the given equations.

Example 28
Solve the equations

$$-3x + y = 18 \qquad (3)$$

$$7x - 3y = -44 \qquad (4)$$

Solution

We modify the equations so that x can be eliminated. For example, by multiplying Equation (3) by 7 and Equation (4) by 3 we find

$$\begin{array}{rcrcr} -21x & + & 7y & = & 126 \\ 21x & - & 9y & = & -132 \end{array}$$

If these equations are now added we can eliminate x. Therefore

$$\begin{array}{rrcrcr} & -21x & + & 7y & = & 126 \\ \text{add} & 21x & - & 9y & = & -132 \\ \hline & 0x & - & 2y & = & -6 \end{array}$$

from which $-2y = -6$, so that $y = 3$. Substituting this value of y into Equation (3) we obtain:

$-3x + 3 = 18$ so that $-3x = 15$ so $x = -5$.

The solution is $x = -5, y = 3$.

Example 29
Solve the equations

$$5x + 3y = -74 \qquad (5)$$

$$-2x - 3y = 26 \qquad (6)$$

Solution

Note that the coefficients of y differ here only in sign.

By adding Equation (5) and Equation (6) we find $3x = -48$ so that $x = -16$.

It then follows that $y = 2$, and the solution is $x = -16, y = 2$.

Task

Solve the equations

$$5x - 7y = -80$$

$$2x + 11y = 106$$

The first step is to modify the equations so that the coefficient of x is the same in both.

If the first is multiplied by 2 then the second equation must be multiplied by what?

Your solution

Answer

5

Write down the resulting equations:

Your solution

Answer

$10x - 14y = -160$, $10x + 55y = 530$

Subtract one equation from the other to eliminate x and hence find y:

Your solution

Answer

$55y - (-14y) = 530 - (-160)$ so $69y = 690$ so $y = 10$.

Now substitute back to find x:

Your solution

Answer

$x = -2$

2. Equations with no solution

On occasions we may encounter a pair of simultaneous equations which have no solution. Consider the following example.

Example 30

Show that the following pair of simultaneous equations have no solution.

$$10x - 2y = -3 \tag{7}$$

$$-5x + y = 1 \tag{8}$$

Solution

Leaving Equation (7) unaltered and multiplying Equation (8) by 2 we find

$$\begin{array}{rcrcr} 10x & - & 2y & = & -3 \\ -10x & + & 2y & = & 2 \end{array}$$

Adding these equations to eliminate x we find that y is eliminated as well:

$$\begin{array}{lrcrcr} & 10x & - & 2y & = & -3 \\ \text{add} & -10x & + & 2y & = & 2 \\ \hline & 0x & + & 0y & = & -1 \end{array}$$

The last line '$0 = -1$' is clearly nonsense.

We say that Equations (7) and (8) are **inconsistent** and they have **no solution**.

3. Equations with an infinite number of solutions

Some pairs of simultaneous equations can possess an infinite number of solutions. Consider the following example.

Example 31

Solve the equations

$$2x + y = 8 \qquad (9)$$

$$4x + 2y = 16 \qquad (10)$$

Solution

If Equation (9) is multiplied by 2 we find both equations are identical: $4x + 2y = 16$. This means that one of them is redundant and we need only consider the single equation

$$2x + y = 8$$

There are infinitely many pairs of values of x and y which satisfy this equation. For example, if $x = 0$ then $y = 8$, if $x = 1$ then $y = 6$, and if $x = -3$ then $y = 14$. We could continue like this producing more and more solutions. Suppose we choose a value, say λ, for x. We can then write

$$2\lambda + y = 8 \qquad \text{so that} \qquad y = 8 - 2\lambda$$

The solution is therefore $x = \lambda$, $y = 8-2\lambda$ **for any value of** λ **whatsoever**. There are an **infinite number of such solutions**.

Exercises

Solve the given simultaneous equations by elimination:

1. (a) $5x + y = 8$, $-3x + 2y = -10$, (b) $2x + 3y = -2$, $5x - 5y = 20$,

   (c) $7x + 11y = -24$, $-9x + y = 46$

2. A straight line has equation of the form $y = ax + b$. The line passes through the points with coordinates $(2, 4)$ and $(-1, 3)$. Write down the simultaneous equations which must be satisfied by a and b. Solve the equations and hence find the equation of the line.

3. A quadratic function $y = ax^2 + bx + c$ is used in signal processing to approximate a more complicated signal. If this function must pass through the points with coordinates $(0, 0)$, $(1, 3)$ and $(5, -11)$ write down the simultaneous equations satisfied by a, b and c. Solve these to find the quadratic function.

Answers

1.(a) $x = 2, y = -2$ (b) $x = 2, y = -2$ (c) $x = -5, y = 1$

2. $y = \frac{1}{3}x + \frac{10}{3}$ 3. $y = -\frac{13}{10}x^2 + \frac{43}{10}x$

4. The graphs of simultaneous linear equations

Each equation in a pair of simultaneous linear equations is, of course, a linear equation and plotting its graph will produce a straight line. The coordinates (x, y) of the point of intersection of the two lines represent the solution of the simultaneous equations because this pair of values satisfies both equations **simultaneously**. If the two lines do not intersect then the equations have no solution (this can only happen if they are distinct and parallel). If the two lines are identical, there are an infinite number of solutions (all points on the line) because the two lines are one on top of the other. Although not the most convenient (or accurate) approach it is possible to solve simultaneous equations using this graphical approach. Consider the following examples.

Example 32

Solve the simultaneous equations

$$4x + y = 9 \qquad (11)$$

$$-x + y - 1 \qquad (12)$$

by plotting two straight line graphs.

Solution

Equation (11) is rearranged into the standard form for the equation of a straight line: $y = -4x + 9$. By selecting two points on the line a graph can be drawn as shown in Figure 8. Similarly, Equation (12) can be rearranged as $y = x - 1$ and its graph drawn. This is also shown in Figure 8.

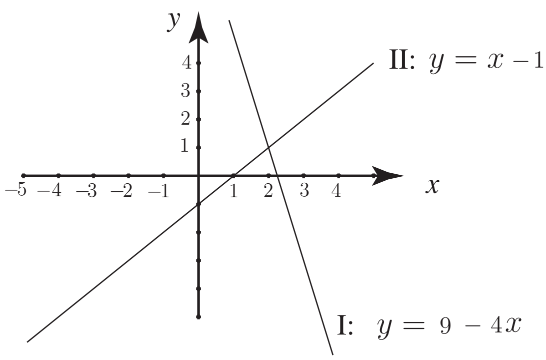


Figure 8: The coordinates of the point of intersection give the required solution

The coordinates of any point on line I satisfy $4x + y = 9$. The coordinates of any point on line II satisfy $-x + y = -1$. At the point where the two lines intersect the x and y coordinates must satisfy both equations simultaneously and so the point of intersection represents the solution. We see from the graph that the point of intersection is $(2, 1)$. The solution of the given equations is therefore $x = 2$, $y = 1$.

Task

Find any solutions of the simultaneous equations: $10x - 2y = 4$, $5x - y = -1$ by graphical method.

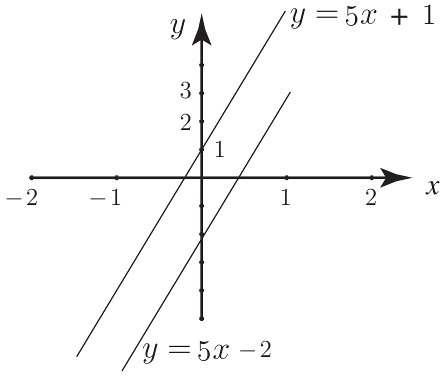
Your solution

Answer

Re-writing the equations in standard form we find

$$y = 5x - 2, \qquad \text{and} \qquad y = 5x + 1$$

Graphs of these lines are shown below. Note that these distinct lines are parallel and so do not intersect. This means that the given simultaneous equations do not have a solution; they are inconsistent.



Exercises

Solve the given equations graphically:

1. $5x - y = 7$, $2x + y = 7$,

2. $2x - 2y = -2$, $5x + y = -9$,

3. $7x + 3y = 25$, $-2x + y = 4$,

4. $4x + 4y = -4$, $x + 7y = -19$.

Answers

1. $x = 2, y = 3$ 2. $x = -5/3, y = -2/3$ 3. $x = 1, y = 6$ 4. $x = 2, y = -3$

Solving Inequalities

3.5

Introduction

An inequality is an expression involving one of the symbols $\geq, \leq, >$ or $<$. This Section will first show how to manipulate inequalities correctly. Then algebraical and graphical methods of solving inequalities will be described.

Prerequisites

Before starting this Section you should ...

- be able to solve linear and quadratic equations

Learning Outcomes

On completion you should be able to ...

- re-arrange expressions involving inequalities
- solve linear and quadratic inequalities

1. The inequality symbols

Recall the definitions of the inequality symbols in Key Point 11:

Key Point 11

The symbols $>$, $<$, $\geq$, $\leq$ are called **inequalities**

$>$ means: 'is greater than', $\geq$ means: 'is greater than or equal to'

$<$ means: 'is less than', $\leq$ means: 'is less than or equal to'

So for example,

$$8 > 7 \qquad 9 \geq 2 \qquad -2 < 3 \qquad 7 \leq 7$$

A number line is often a helpful way of picturing inequalities. Given two numbers a and b, if $b > a$ then b will be to the right of a on the number line as shown in Figure 9.

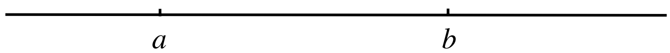


Figure 9: When $b > a$, b is to the right of a on the number line.

Note from Figure 10 that $-3 > -5$, $4 > -2$ and $8 > 5$.

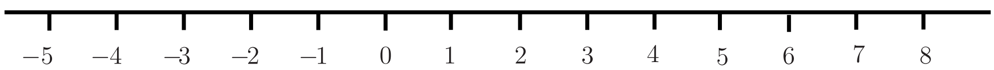


Figure 10

Inequalities can always be written in two ways. For example in English we can state that 8 is greater than 7, or equivalently, that 7 is less than 8. Mathematically we write $8 > 7$ or $7 < 8$. In general if $b > a$ then $a < b$. If $a < b$ then a will be to the left of b on the number line.

Example 33

Rewrite the inequality $-\frac{2}{5} < x$ using only the 'greater than' sign, $>$.

Solution

$-\frac{2}{5} < x$ can be written as $x > -\frac{2}{5}$

Example 34

Rewrite the inequality $5 > x$ using only the 'less than' sign, $<$.

Solution

$5 > x$ can be written as $x < 5$.

Sometimes two inequalities are combined into a single statement. Consider for example the statement $3 < x < 6$. This is a compact way of writing '$3 < x$ **and** $x < 6$'. Now $3 < x$ is equivalent to $x > 3$ and so $3 < x < 6$ means x is greater than 3 but less than 6.
Inequalities obey simple rules when used in conjunction with arithmetical operations:

Key Point 12

1. Adding or subtracting the same quantity from both sides of an inequality leaves the inequality symbol unchanged.
2. Multiplying or dividing both sides by a **positive** number leaves the inequality unchanged.
3. Multiplying or dividing both sides by a **negative** number reverses the inequality.

For example, since $8 > 5$, by adding k to both sides we can state

$$8 + k > 5 + k$$

for any value of k. For example (with $k = -3$) $8 - 3 > 5 - 3$. Further, by multiplying both sides of $8 > 5$ by k we can state $8k > 5k$ provided k is positive. However, $8k < 5k$ if k is negative.

We emphasise that the inequality sign is reversed when multiplying both sides by a negative number. A common mistake is to forget to reverse the inequality symbol. For example if $8 > 5$, multiplying both sides by -1 gives $-8 < -5$.

Task

Find the result of multiplying both sides of the inequality $-18 < 9$ by -3.

Your solution

Answer

$54 > -27$

The **modulus** or **magnitude** sign is sometimes used with inequalities. For example $|x| < 1$ represents the set of all numbers whose actual size, irrespective of sign, is less than 1. This means any value between -1 and 1. Thus

$|x| < 1$ means $-1 < x < 1$

Similarly $|x| > 4$ means all numbers whose size, irrespective of sign, is greater than 4. This means any value greater than 4 or less than -4. Thus

$|x| > 4$ means $x > 4$ or $x < -4$

In general, if k is a positive number:

Key Point 13

$|x| < k$ means $-k < x < k$

$|x| > k$ means $x > k$ or $x < -k$

Exercises

1. State which of the following statements are true and which are false.

(a) $4 > 9$, (b) $4 > 4$, (c) $4 \geq 4$, (d) $0.001 < 10^{-5}$, (e) $|-19| < 100$,
(f) $|-19| > -20$, (g) $0.001 \leq 10^{-3}$

In questions 2-9 rewrite each of the statements without using a modulus sign:

2. $|x| < 2$, 3. $|x| < 5$, 4. $|x| \leq 7.5$, 5. $|x - 3| < 2$,
6. $|x - a| < 1$, 7. $|x| > 2$, 8. $|x| > 7.5$, 9. $|x| \geq 0$.

Answers

1. (a) F (b) F (c) T (d) F (e) T (f) T (g) T

2. $-2 < x < 2$ 3. $-5 < x < 5$ 4. $-7.5 \leq x \leq 7.5$

5. $-2 < x - 3 < 2$ 6. $-1 < x - a < 1$ 7. $x > 2$ or $x < -2$

8. $x > 7.5$ or $x < -7.5$ 9. $x \geq 0$ or $x \leq 0$, in fact any x.

2. Solving linear inequalities algebraically

When we are asked to **solve an inequality**, the inequality will contain an unknown variable, say x. Solving means obtaining all values of x for which the inequality is true. In a **linear inequality** the unknown appears only to the first power, that is as x, and not as x^2, x^3, $x^{1/2}$ and so on.
Consider the following examples.

Example 35

Solve the inequality $4x + 3 > 0$.

Solution

$$\begin{aligned} 4x + 3 &> 0 \\ 4x &> -3, && \text{by subtracting 3 from both sides} \\ x &> -\frac{3}{4} && \text{by dividing both sides by 4.} \end{aligned}$$

Hence all values of x greater than $-\frac{3}{4}$ satisfy $4x + 3 > 0$.

Example 36

Solve the inequality $-3x - 7 \leq 0$.

Solution

$$\begin{aligned} -3x - 7 &\leq 0 \\ -3x &\leq 7 && \text{by adding 7 to both sides} \\ x &\geq -\frac{7}{3} && \text{dividing both sides by } -3 \text{ and } \textbf{reversing the inequality} \end{aligned}$$

Hence all values of x greater than or equal to $-\frac{7}{3}$ satisfy $-3x - 7 \leq 0$.

Task

Solve the inequality $17x + 2 < 4x + 1$.
This is done by making x the subject and obtain it on its own on the left-hand side.

Start by subtracting $4x$ from both sides to remove quantities involving x from the right:

Your solution

Answer
$13x + 2 < 1$

Now subtract 2 from both sides to remove the 2 on the left:

Your solution

Answer
$13x < -1$. Finally, the range of values of x are $x < -1/13$

Example 37

Solve the inequality $|5x - 2| < 4$ and depict the solution graphically.

Solution

$|5x - 2| < 4$ is equivalent to $-4 < 5x - 2 < 4$

We treat each part of the inequality separately:

$$\begin{aligned} -4 &< 5x - 2 && \\ -2 &< 5x && \text{by adding 2 to both sides} \\ -\frac{2}{5} &< x && \text{by dividing both sides by 5} \end{aligned}$$

So $x > -\frac{2}{5}$. Now consider the second part: $5x - 2 < 4$.

$$\begin{aligned} 5x - 2 &< 4 && \\ 5x &< 6 && \text{by adding 2 to both sides} \\ x &< \frac{6}{5} && \text{by dividing both sides by 5} \end{aligned}$$

So $x < \dfrac{6}{5}$.

Solution (contd.)

Putting both parts of the solution together we see that the inequality is satisfied when $-\frac{2}{5} < x < \frac{6}{5}$. This range of values is shown in Figure 11.

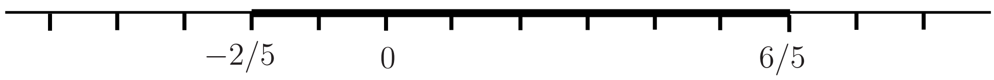


Figure 11: $|5x - 2| < 4$ which is equivalent to $\frac{2}{5} < x < \frac{6}{5}$

Task

Solve the inequality $|1 - 2x| < 5$.

First of all rewrite the inequality without using the modulus sign:

Your solution

$|1 - 2x| < 5$ is equivalent to:

Answer

$-5 < 1 - 2x < 5$

Then treat each part separately. First of all consider $-5 < 1 - 2x$. Solve this:

Your solution

Answer

$x < 3$

The second part is $1 - 2x < 5$. Solve this.

Your solution

Answer

$x > -2$

Finally, give the solution as one statement:

Your solution

Answer

$-2 < x < 3$.

Exercises

In the following questions solve the given inequality algebraically.

1. $4x > 8$	2. $5x > 8$	3. $8x > 5$	4. $8x \leq 5$
5. $2x > 1$	6. $3x < -1$	7. $5x > 2$	8. $2x > 0$
9. $8x < 0$	10. $3x \geq 0$	11. $3x > 4$	12. $\frac{3}{4}x > 1$
13. $4x \leq -3$	14. $3x \leq -4$	15. $5x \geq 0$	16. $4x \leq 0$

17. $5x + 1 < 8$	18. $5x + 1 \leq 8$	19. $7x + 3 \geq 0$
20. $18x + 2 > 9$	21. $14x + 11 > 22$	22. $1 - 5x \leq 0$
23. $2 + 5x \geq 1$	24. $11 - 7x < 2$	25. $5 + 4x > 2x + 1$
26. $\lvert 7x - 3\rvert > 1$	27. $\lvert 2x + 1\rvert \geq 3$	28. $\lvert 5x\rvert < 1$
29. $\lvert 5x\rvert \leq 0$	30. $\lvert 1 - 5x\rvert > 2$	31. $\lvert 2 - 5x\rvert \geq 3$

Answers

1. $x > 2$	2. $x > 8/5$	3. $x > 5/8$	4. $x \leq 5/8$
5. $x > 1/2$	6. $x < -1/3$	7. $x > 2/5$	8. $x > 0$
9. $x < 0$	10. $x \geq 0$	11. $x > 4/3$	12. $x > 4/3$
13. $x \leq -3/4$	14. $x \leq -4/3$	15. $x \geq 0$	16. $x \leq 0$
17. $x < 7/5$	18. $x \leq 7/5$	19. $x \geq -3/7$	20. $x > 7/18$
21. $x > 11/14$	22. $x \geq 1/5$	23. $x \geq -1/5$	24. $x > 9/7$
25. $x > -2$	26. $x > 4/7$ or $x < 2/7$	27. $x \geq 1$ or $x \leq -2$	28. $-1/5 < x < 1/5$
29. $x = 0$	30. $x < -1/5$, $x > 3/5$	31. $x \leq -1/5$, $x \geq 1$	

3. Solving inequalities using graphs

Graphs can be used to help solve inequalities. This approach is particularly useful if the inequality is not linear as, in these cases solving the inequalities algebraically can often be very tricky. Graphics calculators or software can save a lot of time and effort here.

Example 38

Solve graphically the inequality $5x + 2 < 0$.

Solution

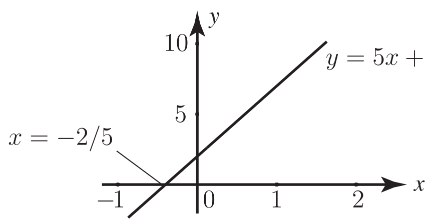


Figure 12: Graph of $y = 5x + 2$.

We consider the function $y = 5x + 2$ whose graph is shown in Figure 12. The values of x which make $5x + 2$ negative are those for which y is negative. We see directly from the graph that y is negative when $x < -\frac{2}{5}$.

Example 39

Find the range of values of x for which $x^2 - x - 6 < 0$.

Solution

We consider the graph of $y = x^2 - x - 6$ which is shown in Figure 13.

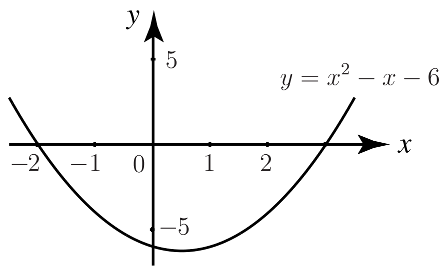


Figure 13: Graph of $y = x^2 - x - 6$

Note that the graph crosses the x axis when $x = -2$ and when $x = 3$, and $x^2 - x - 6$ will be negative when y is negative. Directly from the graph we see that y is negative when $-2 < x < 3$.

Task

Find the range of values of x for which $x^2 - x - 6 > 0$.
The graph of $y = x^2 - x - 6$ has been drawn in Figure 13. We require $y = x^2 - x - 6$ to be positive.

Use the graph to solve the problem:

Your solution

Answer

$x < -2$ or $x > 3$

Example 40

By plotting a graph of $y = 20x^4 - 4x^3 - 143x^2 + 46x + 165$ find the range of values of x for which

$$20x^4 - 4x^3 - 143x^2 + 46x + 165 < 0$$

Solution

A software package has been used to plot the graph which is shown in Figure 14. We see that y is negative when $-2.5 < x < -1$ and is also negative when $1.5 < x < 2.2$.

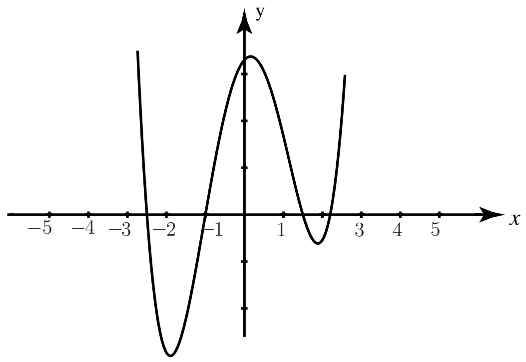


Figure 14: Graph of $y = 20x^4 - 4x^3 - 143x^2 + 46x + 165$

Exercises

In questions 1-5 solve the given inequality graphically:

1. $3x + 1 < 0$ 2. $2x - 7 < 0$ 3. $6x + 9 > 0$, 4. $5x - 3 > 0$ 5. $x^2 - x - 6 < 0$

Answers

1. $x < -1/3$ 2. $x < 7/2$, 3. $x > -3/2$ 4. $x > 3/5$ 5. $-2 < x < 3$

Partial Fractions

3.6

Introduction

It is often helpful to break down a complicated algebraic fraction into a sum of simpler fractions. For example it can be shown that $\dfrac{4x+7}{x^2+3x+2}$ has the same value as $\dfrac{1}{x+2}+\dfrac{3}{x+1}$ for any value of x. We say that

$$\frac{4x+7}{x^2+3x+2} \quad \text{is identically equal to} \quad \frac{1}{x+2}+\frac{3}{x+1}$$

and that the **partial fractions** of $\dfrac{4x+7}{x^2+3x+2}$ are $\dfrac{1}{x+2}$ and $\dfrac{3}{x+1}$.

The ability to express a fraction as its partial fractions is particularly useful in the study of Laplace transforms, Z-transforms, Control Theory and Integration. In this Section we explain how partial fractions are found.

Prerequisites

Before starting this Section you should ...

- be familiar with addition, subtraction, multiplication and division of algebraic fractions

Learning Outcomes

On completion you should be able to ...

- distinguish between proper and improper fractions
- express an algebraic fraction as the sum of its partial fractions

1. Proper and improper fractions

Frequently we find that an algebraic fraction appears in the form

$$\text{algebraic fraction} = \frac{\text{numerator}}{\text{denominator}}$$

where both numerator and denominator are polynomials. For example

$$\frac{x^3+x^2+3x+7}{x^2+1}, \qquad \frac{3x^2-2x+5}{x^2-7x+2}, \qquad \text{and} \qquad \frac{x}{x^4+1},$$

The **degree** of the numerator, n say, is the highest power occurring in the numerator. The degree of the denominator, d say, is the highest power occurring in the denominator. If $d > n$ the fraction is said to be **proper**; the third expression above is such an example. If $d \leq n$ the fraction is said to be **improper**; the first and second expressions above are examples of this type. Before calculating the partial fractions of an algebraic fraction it is important to decide whether the fraction is proper or improper.

Task

For each of the following fractions state the degree of the numerator $(= n)$ and the degree of the denominator $(= d)$. Hence classify the fractions as proper or improper.

(a) $\dfrac{x^3+x^2+3x+7}{x^2+1}$, (b) $\dfrac{3x^2-2x+5}{x^2-7x+2}$, (c) $\dfrac{x}{x^4+1}$, (d) $\dfrac{s^2+4s+5}{(s^2+2s+4)(s+3)}$

(a) Find the degree of denominator and numerator and hence classify (a):

Your solution

Answer

The degree of the numerator, n, is 3. The degree of the denominator, d, is 2.

Because $d \leq n$ the fraction is improper.

(b) Here $n = 2$ and $d = 2$. State whether (b) is proper or improper:

Your solution

Answer

$d \leq n$; the fraction is improper.

(c) Noting that $x = x^1$, state whether (c) is proper or improper:

Your solution

Answer

$d > n$; the fraction is proper.

(d) Find the degree of the numerator and denominator of (d):

Your solution

Answer
Removing the brackets in the denominator we see that $d = 3$. As $n = 2$ this fraction is proper.

Exercise

For each fraction state the degrees of the numerator and denominator, and hence determine which are proper and which are improper.

(a) $\dfrac{x+1}{x}$, (b) $\dfrac{x^2}{x^3 - x}$, (c) $\dfrac{(x-1)(x-2)(x-3)}{x-5}$

Answers (a) $n = 1, d = 1$, improper, (b) $n = 2, d = 3$, proper, (c) $n = 3, d = 1$, improper.

The denominator of an algebraic fraction can often be factorised into a product of linear and/or quadratic factors. Before we can separate algebraic fractions into simpler (partial) fractions we need to completely factorise the denominators into linear and quadratic factors. Linear factors are those of the form $ax + b$; for example $2x + 7$, $3x - 2$ and $4 - x$. **Irreducible quadratic factors** are those of the form $ax^2 + bx + c$ such as $x^2 + x + 1$, and $4x^2 - 2x + 3$, which cannot be factorised into linear factors (these are quadratics with complex roots).

2. Proper fractions with linear factors

Firstly we describe how to calculate partial fractions for proper fractions where the denominator may be written as a product of linear factors. The steps are as follows:

- Factorise the denominator.
- Each factor will produce a partial fraction. A factor such as $3x + 2$ will produce a partial fraction of the form $\dfrac{A}{3x+2}$ where A is an unknown constant. In general a linear factor $ax + b$ will produce a partial fraction $\dfrac{A}{ax+b}$. The unknown constants for each partial fraction may be different and so we will call them A, B, C and so on.
- Evaluate the unknown constants by equating coefficients or using specific values of x.

The sum of the partial fractions is identical to the original algebraic fraction for all values of x.

Key Point 14

A linear factor $ax + b$ in the denominator gives rise to a single partial fraction of the form $\dfrac{A}{ax+b}$

The steps involved in expressing a proper fraction as partial fractions are illustrated in the following Example.

Example 41

Express $\dfrac{7x+10}{2x^2+5x+3}$ in terms of partial fractions.

Solution

Note that this fraction is proper. The denominator is factorised to give $(2x+3)(x+1)$. Each of the linear factors produces a partial fraction. The factor $2x+3$ produces a partial fraction of the form $\dfrac{A}{2x+3}$ and the factor $x+1$ produces a partial fraction $\dfrac{B}{x+1}$, where A and B are constants which we need to find. We write

$$\frac{7x+10}{(2x+3)(x+1)} = \frac{A}{2x+3} + \frac{B}{x+1}$$

By multiplying both sides by $(2x+3)(x+1)$ we obtain

$$7x+10 = A(x+1) + B(2x+3) \qquad \ldots(*)$$

We may now let x take **any value we choose**. By an appropriate choice we can simplify the right-hand side. Let $x = -1$ because this choice eliminates A. We find

$$\begin{aligned} 7(-1)+10 &= A(0) + B(-2+3) \\ 3 &= B \end{aligned}$$

so that the constant B must equal 3. The constant A can be found either by substituting some other value for x or alternatively by 'equating coefficients'.

Observe that, by rearranging the right-hand side, Equation (*) can be written as

$$7x+10 = (A+2B)x + (A+3B)$$

Comparing the coefficients of x on both sides we see that $7 = A+2B$. We already know $B = 3$ and so

$$\begin{aligned} 7 &= A + 2(3) \\ &= A + 6 \end{aligned}$$

from which $A = 1$. We can therefore write

$$\frac{7x+10}{2x^2+5x+3} = \frac{1}{2x+3} + \frac{3}{x+1}$$

We have succeeded in expressing the given fraction as the sum of its partial fractions. The result can always be checked by adding the fractions on the right.

Task

Express $\dfrac{9-4x}{3x^2-x-2}$ in partial fractions.

First factorise the denominator:

Your solution

$3x^2 - x - 2 =$

Answer

$(3x+2)(x-1)$

Because there are two linear factors we write

$$\frac{9-4x}{3x^2-x-2} = \frac{A}{3x+2} + \frac{B}{x-1}$$

Multiply both sides by $(3x+2)(x-1)$ to obtain the equation from which to find A and B:

Your solution

$9 - 4x =$

Answer

$9 - 4x = A(x-1) + B(3x+2)$

Substitute an appropriate value for x to obtain B:

Your solution

Answer

Substitute $x = 1$ and get $B = 1$

Equating coefficients of x to obtain the value of A:

Your solution

Answer

$-4 = A + 3B$, $A = -7$ since $B = 1$

Finally, write down the partial fractions:

Your solution

$$\frac{9-4x}{3x^2-x-2} =$$

Answer

$$\frac{-7}{3x+2} + \frac{1}{x-1}$$

Exercises

1. Find the partial fractions of (a) $\dfrac{5x-1}{(x+1)(x-2)}$, (b) $\dfrac{7x+25}{(x+4)(x+3)}$, (c) $\dfrac{11x+1}{(x-1)(2x+1)}$.

   Check by adding the partial fractions together again.

2. Express each of the following as the sum of partial fractions:

   (a) $\dfrac{3}{(x+1)(x+2)}$, (b) $\dfrac{5}{x^2+7x+12}$, (c) $\dfrac{-3}{(2x+1)(x-3)}$,

Answers

1(a) $\dfrac{2}{x+1}+\dfrac{3}{x-2}$, 1(b) $\dfrac{3}{x+4}+\dfrac{4}{x+3}$ 1(c) $\dfrac{4}{x-1}+\dfrac{3}{2x+1}$,

2(a) $\dfrac{3}{x+1}-\dfrac{3}{x+2}$, 2(b) $\dfrac{5}{x+3}-\dfrac{5}{x+4}$, 2(c) $\dfrac{6}{7(2x+1)}-\dfrac{3}{7(x-3)}$.

3. Proper fractions with repeated linear factors

Sometimes a linear factor appears more than once. For example in

$$\frac{1}{x^2+2x+1}=\frac{1}{(x+1)(x+1)} \quad \text{which equals} \quad \frac{1}{(x+1)^2}$$

the factor $(x+1)$ occurs twice. We call it a **repeated linear factor**. The repeated linear factor $(x+1)^2$ produces two partial fractions of the form $\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}$. In general, a repeated linear factor of the form $(ax+b)^2$ generates two partial fractions of the form

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$$

This is reasonable since the sum of two such fractions **always** gives rise to a **proper** fraction:

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}=\frac{A(ax+b)}{(ax+b)^2}+\frac{B}{(ax+b)^2}=\frac{x(Aa)+Ab+B}{(ax+b)^2}$$

Key Point 15

A repeated linear factor $(ax+b)^2$ in the denominator produces two partial fractions:

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$$

Once again the unknown constants are found by either equating coefficients and/or substituting specific values for x.

Task Express $\dfrac{10x+18}{4x^2+12x+9}$ in partial fractions.

First factorise the denominator:

Your solution

$4x^2+12x+9=$

Answer

$(2x+3)(2x+3)=(2x+3)^2$

There is a repeated linear factor $(2x+3)$ which gives rise to two partial fractions of the form

$$\frac{10x+18}{(2x+3)^2}=\frac{A}{2x+3}+\frac{B}{(2x+3)^2}$$

Multiply both sides through by $(2x+3)^2$ to obtain the equation to be solved to find A and B:

Your solution

Answer

$10x+18=A(2x+3)+B$

Now evaluate the constants A and B by equating coefficients:

Your solution

Answer

Equating the x coefficients gives $10=2A$ so $A=5$. Equating constant terms gives $18=3A+B$ from which $B=3$.

Finally express the answer in partial fractions:

Your solution

Answer

$$\frac{10x+18}{(2x+3)^2}=\frac{5}{2x+3}+\frac{3}{(2x+3)^2}$$

Exercises

Express the following in partial fractions.

(a) $\dfrac{3-x}{x^2-2x+1}$, (b) $-\dfrac{7x-15}{(x-1)^2}$ (c) $\dfrac{3x+14}{x^2+8x+16}$

(d) $\dfrac{5x+18}{(x+4)^2}$ (e) $\dfrac{2x^2-x+1}{(x+1)(x-1)^2}$ (f) $\dfrac{5x^2+23x+24}{(2x+3)(x+2)^2}$

(g) $\dfrac{6x^2-30x+25}{(3x-2)^2(x+7)}$ (h) $\dfrac{s+2}{(s+1)^2}$ (i) $\dfrac{2s+3}{s^2}$.

Answers

(a) $-\dfrac{1}{x-1}+\dfrac{2}{(x-1)^2}$ (b) $-\dfrac{7}{x-1}+\dfrac{8}{(x-1)^2}$ (c) $\dfrac{3}{x+4}+\dfrac{2}{(x+4)^2}$

(d) $\dfrac{5}{x+4}-\dfrac{2}{(x+4)^2}$ (e) $\dfrac{1}{x+1}+\dfrac{1}{x-1}+\dfrac{1}{(x-1)^2}$ (f) $\dfrac{3}{2x+3}+\dfrac{1}{x+2}+\dfrac{2}{(x+2)^2}$

(g) $-\dfrac{1}{3x-2}+\dfrac{1}{(3x-2)^2}+\dfrac{1}{x+7}$ (h) $\dfrac{1}{s+1}+\dfrac{1}{(s+1)^2}$ (i) $\dfrac{2}{s}+\dfrac{3}{s^2}$.

4. Proper fractions with quadratic factors

Sometimes when a denominator is factorised it produces a quadratic term which cannot be factorised into linear factors. One such quadratic factor is x^2+x+1. This factor produces a partial fraction of the form $\dfrac{Ax+B}{x^2+x+1}$. In general a quadratic factor of the form ax^2+bx+c produces a single partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$.

Key Point 16

A quadratic factor ax^2+bx+c in the denominator produces a partial fraction of the form

$$\frac{Ax+B}{ax^2+bx+c}$$

Task

Express as partial fractions $\dfrac{3x+1}{(x^2+x+10)(x-1)}$

Note that the quadratic factor cannot be factorised further. We have

$$\frac{3x+1}{(x^2+x+10)(x-1)} = \frac{Ax+B}{x^2+x+10} + \frac{C}{x-1}$$

First multiply both sides by $(x^2+x+10)(x-1)$:

Your solution

$3x+1 =$

Answer

$(Ax+B)(x-1) + C(x^2+x+10)$

Evaluate C by letting $x = 1$:

Your solution

Answer

$4 = 12C$ so that $C = \dfrac{1}{3}$

Equate coefficients of x^2 and hence find A, and then substitute any other value for x (or equate coefficients of x) to find B:

Your solution

$A =$ $\qquad\qquad$ $B =$

Answer

$-\frac{1}{3}$, $\dfrac{7}{3}$.

Finally express in partial fractions:

Your solution

Answer

$$\frac{3x+1}{(x^2+x+10)(x-1)} = \frac{-\frac{1}{3}x+\frac{7}{3}}{x^2+x+10} + \frac{\frac{1}{3}}{x-1} = \frac{7-x}{3(x^2+x+10)} + \frac{1}{3(x-1)}$$

Engineering Example 3

Admittance

Admittance, Y, is a quantity which is used in analysing electronic circuits. A typical expression for admittance is

$$Y(s) = \frac{s^2+4s+5}{(s^2+2s+4)(s+3)}$$

where s can be thought of as representing frequency. To predict the behaviour of the circuit it is often necessary to express the admittance as the sum of its partial fractions and find the effect of each part separately. Express $Y(s)$ in partial fractions.

The fraction is proper. The denominator contains an irreducible quadratic factor, which cannot be factorised further, and also a linear factor. Thus

$$\frac{s^2+4s+5}{(s^2+2s+4)(s+3)} = \frac{As+B}{s^2+2s+4} + \frac{C}{s+3} \qquad (1)$$

Multiplying both sides of Equation (1) by $(s^2+2s+4)(s+3)$ we obtain

$$s^2+4s+5 = (As+B)(s+3) + C(s^2+2s+4) \qquad (2)$$

To find the constant C we let $s=-3$ in Equation (2) to eliminate A and B.

Thus

$$(-3)^2+4(-3)+5 = C((-3)^2+2(-3)+4)$$

so that

$$2 = 7C \qquad \text{and so} \qquad C = \frac{2}{7}$$

Equating coefficients of s^2 in Equation (2) we find

$$1 = A + C$$

so that $A = 1 - C = 1 - \frac{2}{7} = \frac{5}{7}$.

Equating constant terms in Equation (2) gives $\quad 5 = 3B + 4C$

so that $\quad 3B = 5 - 4C = 5 - 4\left(\frac{2}{7}\right) = \frac{27}{7}$

so $\quad B = \frac{9}{7}$

Finally $\quad Y(s) = \frac{s^2+4s+5}{(s^2+2s+4)(s+3)} = \frac{\frac{5}{7}s+\frac{9}{7}}{s^2+2s+4} + \frac{\frac{2}{7}}{s+3}$

which can be written as $\quad Y(s) = \frac{5s+9}{7(s^2+2s+4)} + \frac{2}{7(s+3)}$

Exercise

Express each of the following as the sum of its partial fractions.

(a) $\dfrac{3}{(x^2+x+1)(x-2)}$, (b) $\dfrac{27x^2-4x+5}{(6x^2+x+2)(x-3)}$, (c) $\dfrac{2x+4}{4x^2+12x+9}$, (d) $\dfrac{6x^2+13x+2}{(x^2+5x+1)(x-1)}$

Answers

(a) $\dfrac{3}{7(x-2)} - \dfrac{3(x+3)}{7(x^2+x+1)}$ (b) $\dfrac{3x+1}{6x^2+x+2} + \dfrac{4}{x-3}$ (c) $\dfrac{1}{2x+3} + \dfrac{1}{(2x+3)^2}$

(d) $\dfrac{3x+1}{x^2+5x+1} + \dfrac{3}{x-1}$.

5. Improper fractions

When calculating the partial fractions of improper fractions an extra polynomial is added to any partial fractions that would normally arise. The added polynomial has degree $n-d$ where n is the degree of the numerator and d is the degree of the denominator. Recall that

a polynomial of degree 0 is a constant, A say,

a polynomial of degree 1 has the form $Ax+B$,

a polynomial of degree 2 has the form Ax^2+Bx+C,

and so on.

If, for example, the improper fraction is such that the numerator has degree 5 and the denominator has degree 3, then $n-d=2$, and we need to add a polynomial of the form Ax^2+Bx+C.

Key Point 17

If a fraction is improper an additional term is included taking the form of a polynomial of degree $n-d$, where n is the degree of the numerator and d is the degree of the denominator.

Example 42

Express as partial fractions

$$\frac{2x^2 - x - 2}{x+1}$$

Solution

The fraction is improper because $n = 2$, $d = 1$ and so $d \leq n$. Here $n - d = 1$, so we need to include as an extra term a polynomial of the form $Bx + C$, in addition to the usual partial fractions. The linear term in the denominator gives rise to a partial fraction $\dfrac{A}{x+1}$. So altogther we have

$$\frac{2x^2 - x - 2}{x+1} = \frac{A}{x+1} + (Bx + C)$$

Multiplying both sides by $x + 1$ we find

$$2x^2 - x - 2 = A + (Bx + C)(x + 1) = Bx^2 + (C + B)x + (C + A)$$

Equating coefficients of x^2 gives $B = 2$.
Equating coefficients of x gives $-1 = C + B$ and so $C = -1 - B = -3$.
Equating the constant terms gives $-2 = C + A$ and so $A = -2 - C = -2 - (-3) = 1$.
Finally, we have

$$\frac{2x^2 - x - 2}{x+1} = \frac{1}{x+1} + 2x - 3$$

Exercise

Express each of the following improper fractions in terms of partial fractions.

(a) $\dfrac{x+3}{x+2}$, (b) $\dfrac{3x-7}{x-3}$, (c) $\dfrac{x^2+2x+2}{x+1}$, (d) $\dfrac{2x^2+7x+7}{x+2}$

(e) $\dfrac{3x^5+4x^4-21x^3-40x^2-24x-29}{(x+2)^2(x-3)}$, (f) $\dfrac{4x^5+8x^4+23x^3+27x^2+25x+9}{(x^2+x+1)(2x+1)}$

Answers

(a) $1 + \dfrac{1}{x+2}$, (b) $3 + \dfrac{2}{x-3}$, (c) $1 + x + \dfrac{1}{x+1}$ (d) $2x + 3 + \dfrac{1}{x+2}$,

(e) $\dfrac{1}{(x+2)^2} + \dfrac{1}{x+2} + \dfrac{1}{x-3} + 3x^2 + x + 2$, (f) $2x^2 + x + 7 + \dfrac{1}{2x+1} + \dfrac{1}{x^2+x+1}$

NOTES

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Workbook 3

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